Brownian Motion and Stochastic Calculus

Exercise Sheet 10

Submit by 12:00 on Wednesday, May 7 via the course homepage.

Exercise 10.1 Let $M \in \mathcal{M}^c_{0,\mathrm{loc}}$. Show that if $E[\langle M \rangle_t] < \infty$ for all $t \geqslant 0$, then M is a continuous square-integrable martingale.

Exercise 10.2 Let $(W_t)_{t\geqslant 0}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$ and $(X_t)_{0\leqslant t\leqslant T}$ the unique strong solution (by Theorem 4.7.4) in the space $\mathcal{R}^2_c$ of the SDE

$$\mathrm{d}X_t = f(X_t)\,\mathrm{d}t + g(X_t)\,\mathrm{d}W_t, \quad X_0 = x_0,$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are Lipschitz-continuous functions and $x_0 \in \mathbb{R}$ is a constant.

(a) Find a non-constant function $\varphi \in C^2(\mathbb{R};\mathbb{R})$ such that the process $Y = (Y_t)_{0\leqslant t\leqslant T}$ given by $Y_t := \varphi(X_t)$ is a local martingale, and derive an SDE for Y (which no longer involves X).

Hint: The general solution of the ODE

$$y' f(x) + \frac{1}{2} y'' g^2(x) = 0$$

is of the form

$$y(x) = a + b \int_0^x \exp\Big(-2 \int_0^u \frac{f(v)}{g^2(v)}\, dv \Big)\, du,$$

where a and b are constants.

(b) Assume additionally that f is negative on $(-\infty, 0)$ and positive on $[0, \infty)$. Show that Y is then a martingale.

Exercise 10.3 Let $W = (W_t)_{t\geqslant 0}$ be a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}, P)$ satisfying the usual conditions. Fix constants $\theta, \sigma, x_0 \in \mathbb{R}$ with $\sigma > 0$.

(a) Find a strong solution to the *Langevin equation*

$$\mathrm{d}X_t = -\theta X_t\,\mathrm{d}t + \sigma\,\mathrm{d}W_t, \quad X_0 = x_0.$$

Hint: Assume first that a strong solution X exists and consider $U_t = e^{\theta t} X_t$.

(b) Let X denote your strong solution from part (a). Show that there exists a Brownian motion B such that $Y := X^2$ satisfies the SDE

$$\mathrm{d}Y_t = (-2\theta Y_t + \sigma^2)\,\mathrm{d}t + 2\sigma\sqrt{Y_t}\,\mathrm{d}B_t. \tag{$*$}$$

In other words, show that $(\Omega, \mathcal{F}, \mathbb{F}, P, B, Y)$ is a weak solution of the SDE $(*)$.

Exercise 10.4 Let $(W_t)_{t\geqslant 0}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$, and consider the SDE

$$\mathrm{d}X_t = \left(\sqrt{1+X_t^2} + \frac{1}{2}X_t\right)\mathrm{d}t \;+\; \sqrt{1+X_t^2}\,\mathrm{d}W_t, \quad X_0 = x_0 \in \mathbb{R}. \tag{$**$}$$

(a) Using Theorem 4.7.4, show that for any $T \in (0,\infty)$ and $x_0 \in \mathbb{R}$, the SDE $(**)$ has on $[0,T]$ a unique strong solution in $\mathcal{R}_c^2$.

(b) Show directly (and without using part (a)) that the process $X = (X_t)_{t\geqslant 0}$ given by $X_t = \sinh(\sinh^{-1}(x_0) + t + W_t)$ is the unique solution of $(**)$.

Hint: The identity $\cosh(\sinh^{-1}(x)) = \sqrt{1+x^2}$ *may be useful..*