Brownian Motion and Stochastic Calculus

Exercise Sheet 10

Submit by 12:00 on Wednesday, May 7 via the course homepage.

Exercise 10.1 Let $M \in \mathcal{M}^c_{0,\mathrm{loc}}$. Show that if $E[\langle M\rangle_t] < \infty$ for all $t \geqslant 0$, then M is a continuous square-integrable martingale.

Solution 10.1 As both M and $M^2 - \langle M\rangle$ are continuous local martingales null at zero, Exercise 8.1(b) says that the sequences of stopping times $(\sigma_n)_{n\in\mathbb{N}}$ and $(\rho_n)_{n\in\mathbb{N}}$ defined by

$$\sigma_n := \inf\{t \geqslant 0 : |M_t| \geqslant n\} \quad \text{and} \quad \rho_n := \inf\{t \geqslant 0 : |M_t^2 - \langle M\rangle_t| \geqslant n\}$$

are localising sequences for M and $M^2 - \langle M\rangle$, respectively. Setting $\tau_n := \sigma_n \wedge \rho_n$ for each $n \in \mathbb{N}$, Exercise 7.1(c) gives that $(\tau_n)_{n\in\mathbb{N}}$ is a localising sequence for both M and $M^2 - \langle M\rangle$. So for each $n \in \mathbb{N}$, the stopped processes M^{τ_n} and $(M^2 - \langle M\rangle)^{\tau_n}$ are continuous bounded martingales. The martingale property gives for each $t \geqslant 0$ and $n \in \mathbb{N}$ that

$$E[M^2_{\tau_n\wedge t}] = E[\langle M\rangle_{\tau_n\wedge t}] \leqslant E[\langle M\rangle_t] =: C(t)$$

where the constant $C(t)$ is finite by assumption. Since $M_{\tau_n\wedge t} \to M_t$ P-a.s., Fatou's lemma gives

$$E[M_t^2] \leqslant \liminf_{n\to\infty} E[M^2_{\tau_n\wedge t}] \leqslant C(t),$$

so that M is a square-integrable process. It remains to establish the martingale property. So fix $0 \leqslant s \leqslant t$ and note that for each $n \in \mathbb{N}$, we have by the martingale property of M^{τ_n} that

$$E[M_{\tau_n\wedge t} \,|\, \mathcal{F}_s] = M_{\tau_n\wedge s}. \tag{1}$$

Since $\sup_{n\in\mathbb{N}} E[M^2_{\tau_n\wedge t}] \leqslant C(t)$, the sequence of random variables $(M_{\tau_n\wedge t})_{n\in\mathbb{N}}$ is bounded in $L^2(P)$ and hence uniformly integrable. So as $M_{\tau_n\wedge t} \to M_t$ P-a.s., we have $M_{\tau_n\wedge t} \to M_t$ in L^1. Now for a σ-field $\mathcal{G}$, the map from $L^1(P)$ to $L^1(P)$ defined by $X \mapsto E[X \,|\, \mathcal{G}]$ is (Lipschitz-) continuous, since for any $X, Y \in L^1(P)$, we have

$$E\Big[\big|E[X \,|\, \mathcal{G}] - E[Y \,|\, \mathcal{G}]\big|\Big] \leqslant E[|X - Y|].$$

It follows that as $n \to \infty$,

$$E[M_{\tau_n\wedge t} \,|\, \mathcal{F}_s] \to E[M_t \,|\, \mathcal{F}_s] \quad \text{in } L^1(P).$$

Analogously, we also have $M_{\tau_n \wedge s} \to M_s$ in $L^1(P)$, and so by taking the limit in $L^1(P)$ of (1) as $n \to \infty$ yields

$$E[M_t \,|\, \mathcal{F}_s] = M_s,$$

completing the proof.

Exercise 10.2 Let $(W_t)_{t \geqslant 0}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$ and $(X_t)_{0 \leqslant t \leqslant T}$ the unique strong solution (by Theorem 4.7.4) in the space $\mathcal{R}_c^2$ of the SDE

$$\mathrm{d}X_t = f(X_t)\,\mathrm{d}t + g(X_t)\,\mathrm{d}W_t, \quad X_0 = x_0,$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are Lipschitz-continuous functions and $x_0 \in \mathbb{R}$ is a constant.

(a) Find a non-constant function $\varphi \in C^2(\mathbb{R};\mathbb{R})$ such that the process $Y = (Y_t)_{0 \leqslant t \leqslant T}$ given by $Y_t := \varphi(X_t)$ is a local martingale, and derive an SDE for Y (which no longer involves X).

Hint: The general solution of the ODE

$$y' f(x) + \frac{1}{2} y'' g^2(x) = 0$$

is of the form

$$y(x) = a + b \int_0^x \exp\Big(-2 \int_0^u \frac{f(v)}{g^2(v)}\, dv \Big)\, du,$$

where a and b are constants.

(b) Assume additionally that f is negative on $(-\infty, 0)$ and positive on $[0, \infty)$. Show that Y is then a martingale.

Solution 10.2

(a) Applying Itô's formula to $Y = \varphi(X)$ for a general $\varphi \in C^2(\mathbb{R};\mathbb{R})$, we obtain that

$$Y_t = \varphi(x_0) + \int_0^t \varphi'(X_s) g(X_s)\,\mathrm{d}W_s + \int_0^t \Big(\varphi'(X_s) f(X_s) + \frac{1}{2} \varphi''(X_s) g^2(X_s) \Big)\,\mathrm{d}s.$$

We thus obtain that Y is a local martingale if φ solves the ODE

$$\varphi'(x) f(x) + \frac{1}{2} \varphi''(x) g^2(x) = 0.$$

So using the hint, by picking any constants $a, b \in \mathbb{R}$ and setting

$$\varphi(x) := a + b \int_0^x \exp\Big(-2 \int_0^u \frac{f(v)}{g^2(v)}\,\mathrm{d}v \Big)\,\mathrm{d}u,$$

we have that Y is a local martingale, as required. As the problem asks φ to be not constant, we require that $b \neq 0$. It now remains to derive the SDE for Y. By our choice of φ, we have

$$\mathrm{d}Y_t = \varphi'(X_t)g(X_t)\,\mathrm{d}W_s, \quad Y_0 = \varphi(x_0).$$

Also, since $b \neq 0$, φ is a continuous bijection from $\mathbb{R}$ onto $\mathbb{R}$, and therefore the inverse function φ^{-1} exists (and is continuous). So we can write $X_t = \varphi^{-1}(Y_t)$, and so we get that Y satisfies the SDE

$$\mathrm{d}Y_t = (\varphi' \circ \varphi^{-1})(Y_t)\,(g \circ \varphi^{-1})(Y_t)\,\mathrm{d}W_t, \quad Y_0 = \varphi(x_0).$$

(b) As $Y - \varphi(x_0) \in \mathcal{M}^c_{0,\mathrm{loc}}$, it suffices by Exercise 10.1 to show that $E[\langle Y\rangle_t] < \infty$ for all $t \geqslant 0$. Now as

$$Y_t = \varphi(x_0) + \int_0^t \varphi'(X_s)g(X_s)\,\mathrm{d}W_s,$$

we have

$$E[\langle Y\rangle_t] = E\bigg[\int_0^t \Big(\varphi'(X_s)g(X_s)\Big)^2\,\mathrm{d}s\bigg],$$

which we should like to show is finite. To this end, we first observe that since f is negative on $(-\infty, 0)$ and positive on $[0,\infty)$, we have from the equality $\phi'(x) = b\exp(-2\int_0^x \frac{f(v)}{g^2(v)}\,\mathrm{d}v)$ that

$$\sup_{x\in\mathbb{R}} |\varphi'(x)| \leqslant |b|.$$

Moreover, as $g : \mathbb{R} \to \mathbb{R}$ is Lipschitz-continuous, there exists a constant $k > 0$ such that

$$|g(x)| \leqslant |g(0)| + k|x|.$$

As $(c+d)^2 \leqslant 2(c^2+d^2)$ for any $c, d \in \mathbb{R}$, we obtain

$$g(x)^2 \leqslant 2g(0)^2 + 2k^2\,x^2,$$

and so

$$\Big(\phi'(x)g(x)\Big)^2 \leqslant 2b^2g(0) + 2b^2k^2x^2 =: \alpha + \beta x^2,$$

where we relabel $\alpha := 2b^2g(0)$ and $\beta := 2b^2k^2$. We thus have that

$$E\bigg[\int_0^T \Big(\varphi'(X_s)g(X_s)\Big)^2\,\mathrm{d}s\bigg] \leqslant \alpha T + \beta E\bigg[\int_0^T X_s^2\,\mathrm{d}s\bigg].$$

Now using that $X \in \mathcal{R}^2_c$, we write

$$E\bigg[\int_0^T X_s^2\,\mathrm{d}s\bigg] \leqslant TE\Big[\sup_{0\leqslant t\leqslant T} X_t^2\Big] < \infty.$$

This completes the proof.

Exercise 10.3 Let $W = (W_t)_{t \geqslant 0}$ be a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geqslant 0}, P)$ satisfying the usual conditions. Fix constants $\theta, \sigma, x_0 \in \mathbb{R}$ with $\sigma > 0$.

(a) Find a strong solution to the *Langevin equation*

$$\mathrm{d}X_t = -\theta X_t \,\mathrm{d}t + \sigma \,\mathrm{d}W_t, \quad X_0 = x_0.$$

Hint: Assume first that a strong solution X exists and consider $U_t = e^{\theta t} X_t$.

(b) Let X denote your strong solution from part (a). Show that there exists a Brownian motion B such that $Y := X^2$ satisfies the SDE

$$\mathrm{d}Y_t = (-2\theta Y_t + \sigma^2)\,\mathrm{d}t + 2\sigma\sqrt{Y_t}\,\mathrm{d}B_t. \tag{$*$}$$

In other words, show that $(\Omega, \mathcal{F}, \mathbb{F}, P, B, Y)$ is a weak solution of the SDE $(*)$.

Solution 10.3

(a) Following the hint, we assume first that a strong solution X exists, and we apply Itô's formula to the process $U = (U_t)_{t \geqslant 0}$ given by $U_t = e^{\theta t} X_t$ to get

$$\mathrm{d}U_t = \theta e^{\theta t} X_t \,\mathrm{d}t + e^{\theta t}(-\theta X_t \,\mathrm{d}t + \sigma \,\mathrm{d}W_t) = \sigma e^{\theta t} \,\mathrm{d}W_t.$$

So we have $U_t = x_0 + \int_0^t \sigma e^{\theta s} \,\mathrm{d}W_s$. In particular, we have written U_t as an expression that does not depend on the process X we had to assume existed in the first place. We now *define* the process X by $X_t := e^{-\theta t} U_t$. Applying Itô's formula again yields

$$\mathrm{d}X_t = -\theta e^{-\theta t} U_t \,\mathrm{d}t + e^{-\theta t} \mathrm{d}U_t = -\theta X_t \,\mathrm{d}t + \sigma \,\mathrm{d}W_t.$$

As $X_0 = U_0 = x_0$, it follows that X is a strong solution to the Langevin equation, as required.

(b) By Itô's formula, we have

$$\begin{aligned}
\mathrm{d}Y_t &= 2X_t(-\theta X_t \,\mathrm{d}t + \sigma \,\mathrm{d}W_t) + \sigma^2 \,\mathrm{d}t \\
&= (-2\theta X_t^2 + \sigma^2)\,\mathrm{d}t + 2\sigma X_t \,\mathrm{d}W_t \\
&= (-2\theta Y_t + \sigma^2)\,\mathrm{d}t + 2\sigma\sqrt{Y_t}\,\mathrm{sign}(X_t)\,\mathrm{d}W_t.
\end{aligned}$$

Defining $B_t := \int_0^t \mathrm{sign}(X_s)\,\mathrm{d}W_s$, we thus have

$$\mathrm{d}Y_t = (-2\theta Y_t + \sigma^2)\,\mathrm{d}t + 2\sigma\sqrt{Y_t}\,\mathrm{d}B_t.$$

But by Lévy's characterisation theorem, we have that B is a Brownian motion, which completes the proof.

Exercise 10.4 Let $(W_t)_{t\geqslant 0}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$, and consider the SDE

$$\mathrm{d}X_t = \left(\sqrt{1+X_t^2} + \frac{1}{2}X_t\right)\mathrm{d}t \;+\; \sqrt{1+X_t^2}\,\mathrm{d}W_t, \quad X_0 = x_0 \in \mathbb{R}. \qquad (**)$$

(a) Using Theorem 4.7.4, show that for any $T \in (0,\infty)$ and $x_0 \in \mathbb{R}$, the SDE $(**)$ has on $[0,T]$ a unique strong solution in $\mathcal{R}_c^2$.

(b) Show directly (and without using part (a)) that the process $X = (X_t)_{t\geqslant 0}$ given by $X_t = \sinh(\sinh^{-1}(x_0) + t + W_t)$ is the unique solution of $(**)$.

Hint: The identity $\cosh(\sinh^{-1}(x)) = \sqrt{1+x^2}$ *may be useful..*

Solution 10.4

(a) We see that $(**)$ is of the form

$$\mathrm{d}X_t = a(X_t)\,\mathrm{d}t + b(X_t)\,\mathrm{d}W_t, \quad X_0 = x_0 \in \mathbb{R},$$

where

$$a(x) = \sqrt{1+x^2} + \frac{1}{2}x \quad \text{and} \quad b(x) = \sqrt{1+x^2}.$$

We observe that

$$\sup_{x\in\mathbb{R}} |b'(x)| = \sup_{x\in\mathbb{R}} \left|\frac{x}{\sqrt{1+x^2}}\right| \leqslant 1$$

and therefore

$$\sup_{x\in\mathbb{R}} |a'(x)| = \sup_{x\in\mathbb{R}} \left|\frac{x}{\sqrt{1+x^2}} + \frac{1}{2}\right| \leqslant \frac{3}{2}.$$

So since a and b have bounded derivatives, it follows from the mean value theorem that a and b are Lipschitz-continuous. Also, since a is Lipschitz-continuous and $|a(x)| \leqslant |a(x) - a(0)| + |a(0)|$, we see that a satisfies the linear growth condition of Theorem 4.7.4, and similarly so does b. We may thus conclude by Theorem 4.7.4 that for any $T \in (0,\infty)$ and $x_0 \in \mathbb{R}$, there exists on $[0,T]$ a unique strong solution to $(**)$.

(b) Set $Z_t := \sinh^{-1}(x_0) + t + W_t$. Then $\mathrm{d}Z_t = \mathrm{d}t + \mathrm{d}W_t$ and $\mathrm{d}\langle Z\rangle_t = \mathrm{d}t$. Now noting that $X_t = \sinh(Z_t)$, we apply Itô's formula to get

$$\begin{aligned}
\mathrm{d}X_t &= \cosh(Z_t)(\mathrm{d}t + \mathrm{d}W_t) + \frac{1}{2}\sinh(Z_t)\,\mathrm{d}t \\
&= \cosh\Big(\sinh^{-1}(X_t)\Big)(\mathrm{d}t + \mathrm{d}W_t) + \frac{1}{2}X_t\,\mathrm{d}t \\
&= \left(\sqrt{1+X_t^2} + \frac{1}{2}X_t\right)\mathrm{d}t + \sqrt{1+X_t^2}\,\mathrm{d}W_t,
\end{aligned}$$

where the last step uses the identity given in the hint. It follows that X is a solution to $(**)$.

Conversely, let X denote any solution to $(**)$. In order to establish uniqueness of the solution to $(**)$, we need to show that $X_t = \sinh(\sinh^{-1}(x_0) + t + W_t)$. To this end, let $f(x) = \sinh^{-1}(x)$ and compute the derivatives

$$f'(x) = \frac{1}{\sqrt{1+x^2}} \quad \text{and} \quad f''(x) = -\frac{x}{(1+x^2)^{3/2}}.$$

Then applying Itô's formula to $f(X_t)$, we get

$$\begin{aligned}
\mathrm{d}f(X_t) &= \frac{1}{\sqrt{1+X_t^2}}\,\mathrm{d}X_t - \frac{X_t}{2(1+X_t^2)^{3/2}}\,\mathrm{d}\langle X\rangle_t \\
&= \left(1 + \frac{X_t}{2\sqrt{1+X_t^2}}\right)\mathrm{d}t + \mathrm{d}W_t - \frac{X_t}{2(1+X_t^2)^{3/2}}(1+X_t^2)\,\mathrm{d}t \\
&= \mathrm{d}t + \mathrm{d}W_t.
\end{aligned}$$

So $f(X_t) = \sinh^{-1}(x_0) + t + W_t$, so that $X_t = \sinh(\sinh^{-1}(x_0) + t + W_t)$. This completes the proof.