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Brownian Motion and Stochastic Calculus

Exercise Sheet 11

Submit by 12:00 on Wednesday, May 14 via the course homepage.

Exercise 11.1 Let $(W_t)_{t\geqslant 0}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. Consider the SDE

$$X_t = \int_0^t b(X_s)\,\mathrm{d}s + \int_0^t \sigma(X_s)\,\mathrm{d}W_s, \quad X_0 = 0, \tag{$*$}$$

where $b(x) := 3x^{1/3}$ and $\sigma(x) := 3x^{2/3}$. Show that $(*)$ has uncountably many strong solutions of the form

$$X_t^{(\theta)} := \begin{cases} 0, & 0 \leqslant t < \beta_\theta, \\ W_t^3, & \beta_\theta \leqslant t < \infty, \end{cases}$$

where $0 \leqslant \theta < \infty$ is any fixed constant and $\beta_\theta := \inf\{s \geqslant \theta : W_s = 0\}$.

Exercise 11.2 Consider the SDE

$$\begin{aligned} \mathrm{d}X_t^x &= a(X_t^x)\,\mathrm{d}t + b(X_t^x)\,\mathrm{d}W_t, \\ X_0^x &= x, \end{aligned} \tag{$**$}$$

where W is an $\mathbb{R}^m$-valued Brownian motion and the functions $a : \mathbb{R}^d \to \mathbb{R}^d$ and $b : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are measurable and locally bounded (that is, they are bounded on compact sets). Let $U \subseteq \mathbb{R}^d$ be a bounded open set with the property that the stopping time $T_U^x := \inf\{s \geqslant 0 : X_s^x \notin U\}$ is P-integrable for all $x \in U$. Consider the boundary problem

$$\begin{aligned} Lu(x) + c(x)u(x) &= -f(x) \qquad \text{for } x \in U, \\ u(x) &= g(x) \qquad \text{for } x \in \partial U, \end{aligned} \tag{$***$}$$

where $c, f \in C_b(U)$ and $g \in C_b(\partial U)$ are given functions such that $c \leqslant 0$ on U, and the linear operator L is defined by

$$Lh(x) := \sum_{i=1}^d a_i(x)\,\frac{\partial h}{\partial x^i}(x) + \frac{1}{2}\sum_{i,j=1}^d (b\,b^\top)_{ij}(x)\,\frac{\partial^2 h}{\partial x^i\,\partial x^j}(x).$$

Suppose that $(X_t^x)_{t\geqslant 0}$ solves the SDE $(**)$ for some $x \in U$ and $u \in C^2(U) \cap C(\overline{U})$ is a solution to the boundary problem $(***)$. Show that

$$u(x) = E\bigg[g(X_{T_U^x}^x)\,\exp\bigg(\int_0^{T_U^x} c(X_s^x)\,\mathrm{d}s\bigg)\bigg] + E\bigg[\int_0^{T_U^x} f(X_s^x)\,\exp\bigg(\int_0^s c(X_r^x)\,\mathrm{d}r\bigg)\,\mathrm{d}s\bigg].$$

Hint: You may use the following standard result from analysis.

Let $K \subseteq \mathbb{R}^d$ be compact and $C \subseteq \mathbb{R}^d$ be closed such that $C \cap K = \emptyset$. Then there exists a smooth function $\psi : \mathbb{R}^d \to \mathbb{R}$ such that $\psi \equiv 1$ on K and $\psi \equiv 0$ on C.

Exercise 11.3 Consider a probability space $(\Omega, \mathcal{F}, P)$ supporting a Brownian motion $W = (W_t)_{t \geqslant 0}$. Denote by $\mathbb{F} = (\mathcal{F}_t)_{t \geqslant 0}$ the P-augmentation of the (raw) filtration generated by W. Let $T > 0$, $\alpha > 0$ and let F be a bounded, $\mathcal{F}_T$-measurable random variable.

(a) Show that the process $X = (X_t)_{0 \leqslant t \leqslant T}$ given by

$$X_t = -\alpha \log E[\exp(-F/\alpha) \,|\, \mathcal{F}_t]$$

satisfies the BSDE

$$\begin{aligned} \mathrm{d}X_t &= \frac{1}{2\alpha} Z_t^2 \,\mathrm{d}t + Z_t \,\mathrm{d}W_t, \\ X_T &= F. \end{aligned}$$

Hint: We have that $X_t = -\alpha \log Y_t$, where $Y_t := E[\exp(-F/\alpha) \,|\, \mathcal{F}_t]$. Apply Itô's representation theorem to Y_T and Itô's formula to X to derive a solution pair $(X, Z) \in \mathcal{R}^2 \times L^2(W)$ for the BSDE.

(b) Let $b \in \mathbb{R}$. Show that the process $X = (X_t)_{0 \leqslant t \leqslant T}$ given by

$$X_t = -\alpha \left(\frac{1}{2} b^2 (t - T) - b\, W_t + \log E[\exp(bW_T - F/\alpha) \,|\, \mathcal{F}_t] \right)$$

satisfies the BSDE

$$\begin{aligned} \mathrm{d}X_t &= \left(\frac{1}{2\alpha}\, Z_t^2 - b\, Z_t \right) \mathrm{d}t + Z_t \,\mathrm{d}W_t, \\ X_T &= F. \end{aligned}$$