Brownian Motion and Stochastic Calculus

Exercise Sheet 11

Submit by 12:00 on Wednesday, May 14 via the course homepage.

Exercise 11.1 Let $(W_t)_{t \geqslant 0}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. Consider the SDE

$$X_t = \int_0^t b(X_s)\,\mathrm{d}s + \int_0^t \sigma(X_s)\,\mathrm{d}W_s, \quad X_0 = 0, \tag{$*$}$$

where $b(x) := 3x^{1/3}$ and $\sigma(x) := 3x^{2/3}$. Show that $(*)$ has uncountably many strong solutions of the form

$$X_t^{(\theta)} := \begin{cases} 0, & 0 \leqslant t < \beta_\theta, \\ W_t^3, & \beta_\theta \leqslant t < \infty, \end{cases}$$

where $0 \leqslant \theta < \infty$ is any fixed constant and $\beta_\theta := \inf\{s \geqslant \theta : W_s = 0\}$.

Solution 11.1 For each $0 \leqslant \theta < \infty$ we define the process $(S_t^{(\theta)})_{t \geqslant 0}$ by

$$S_t^{(\theta)} = \begin{cases} 0, & 0 \leqslant t < \beta_\theta, \\ W_t, & \beta_\theta \leqslant t < \infty, \end{cases}$$

where $\beta_\theta := \inf\{s \geqslant \theta : W_s = 0\}$ is a stopping time. Now from Exercise 7.1(b), we know that a stopped local martingale is again a local martingale, from which we can conclude that also a stopped semimartingale is again a semimartingale. We also easily see that the difference of two semimartingales is again a semimartingale. So since $W_{\beta_\theta} = 0$, we have that

$$S^{(\theta)} = W - W^{\beta_\theta}$$

and therefore that $S^{(\theta)}$ is a semimartingale. Moreover, we have

$$\begin{aligned} \langle S^{(\theta)} \rangle_t &= \langle W \rangle_t - 2\langle W, W^{\beta_\theta} \rangle_t + \langle W^{\beta_\theta} \rangle_t \\ &= \langle W \rangle_t - 2\langle W \rangle_t^{\beta_\theta} + \langle W \rangle_t^{\beta_\theta} \\ &= t - t \wedge \beta_\theta \\ &= (t - \beta_\theta)\mathbf{1}_{\{t \geqslant \beta_\theta\}}. \end{aligned}$$

So on $\{t \geqslant \beta_\theta\}$ we have that $\mathrm{d}S_t^{(\theta)} = \mathrm{d}W_t$ and that $\mathrm{d}\langle S^{(\theta)} \rangle_t = \mathrm{d}t$. So applying Itô's formula to the process $X_t^{(\theta)} := f(S_t^{(\theta)})$ for $f(x) := x^3$, we get that

$$\begin{aligned}
X_t^{(\theta)} &= (S_0^{(\theta)})^3 + \int_0^t 3(S_s^{(\theta)})^2 \, \mathrm{d}S_s^{(\theta)} + 3\int_0^t S_s^{(\theta)} \, \mathrm{d}\langle S^{(\theta)}\rangle_s \\
&= \int_0^t 3(X_s^{(\theta)})^{2/3} \, \mathrm{d}S_s^{(\theta)} + 3\int_0^t (X_s^{(\theta)})^{1/3} \, \mathrm{d}\langle S^{(\theta)}\rangle_s \\
&= \int_{\beta_\theta}^{t\vee\beta_\theta} 3(X_s^{(\theta)})^{2/3} \, \mathrm{d}S_s^{(\theta)} + 3\int_{\beta_\theta}^{t\vee\beta_\theta} (X_s^{(\theta)})^{1/3} \, \mathrm{d}\langle S^{(\theta)}\rangle_s \\
&= \int_{\beta_\theta}^{t\vee\beta_\theta} 3(X_s^{(\theta)})^{2/3} \, \mathrm{d}W_s + 3\int_{\beta_\theta}^{t\vee\beta_\theta} (X_s^{(\theta)})^{1/3} \, \mathrm{d}s \\
&= \int_0^t 3(X_s^{(\theta)})^{2/3} \, \mathrm{d}W_s + 3\int_0^t (X_s^{(\theta)})^{1/3} \, \mathrm{d}s,
\end{aligned}$$

where in third and fifth steps we use that $X_s^{(\theta)} = 0$ on for all $s < \beta_\theta$. We have thus shown that $X^{(\theta)}$ is a strong solution to $(*)$ for each $0 \leqslant \theta < \infty$, as required.

Exercise 11.2 Consider the SDE

$$\begin{aligned}
\mathrm{d}X_t^x &= a(X_t^x)\, \mathrm{d}t + b(X_t^x)\, \mathrm{d}W_t, \\
X_0^x &= x,
\end{aligned} \tag{$**$}$$

where W is an $\mathbb{R}^m$-valued Brownian motion and the functions $a : \mathbb{R}^d \to \mathbb{R}^d$ and $b : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are measurable and locally bounded (that is, they are bounded on compact sets). Let $U \subseteq \mathbb{R}^d$ be a bounded open set with the property that the stopping time $T_U^x := \inf\{s \geqslant 0 : X_s^x \notin U\}$ is P-integrable for all $x \in U$. Consider the boundary problem

$$\begin{aligned}
Lu(x) + c(x)u(x) &= -f(x) \quad \text{for } x \in U, \\
u(x) &= g(x) \quad \text{for } x \in \partial U,
\end{aligned} \tag{$***$}$$

where $c, f \in C_b(U)$ and $g \in C_b(\partial U)$ are given functions such that $c \leqslant 0$ on U, and the linear operator L is defined by

$$Lh(x) := \sum_{i=1}^d a_i(x) \frac{\partial h}{\partial x^i}(x) + \frac{1}{2}\sum_{i,j=1}^d (b\, b^\top)_{ij}(x) \frac{\partial^2 h}{\partial x^i \, \partial x^j}(x).$$

Suppose that $(X_t^x)_{t\geqslant 0}$ solves the SDE $(**)$ for some $x \in U$ and $u \in C^2(U) \cap C(\overline{U})$ is a solution to the boundary problem $(***)$. Show that

$$u(x) = E\bigg[g(X_{T_U^x}^x)\, \exp\bigg(\int_0^{T_U^x} c(X_s^x)\, \mathrm{d}s\bigg)\bigg] + E\bigg[\int_0^{T_U^x} f(X_s^x)\, \exp\bigg(\int_0^s c(X_r^x)\, \mathrm{d}r\bigg)\, \mathrm{d}s\bigg].$$

Hint: You may use the following standard result from analysis.

Let $K \subseteq \mathbb{R}^d$ be compact and $C \subseteq \mathbb{R}^d$ be closed such that $C \cap K = \emptyset$. Then there exists a smooth function $\psi : \mathbb{R}^d \to \mathbb{R}$ such that $\psi \equiv 1$ on K and $\psi \equiv 0$ on C.

Solution 11.2 Fix $x \in U$. For each $n \in \mathbb{N}$ with $\operatorname{dist}(x, U^c) > 1/n$, define the stopping time

$$T_n^x := \inf\left\{s \geqslant 0 : \operatorname{dist}(X_s^x, U^c) \leqslant \frac{1}{n}\right\} \leqslant T_U^x.$$

For each $n \in \mathbb{N}$, we define the sets

$$\begin{aligned} K_n &:= \{x \in \mathbb{R}^d : \operatorname{dist}(x, U^c) \geqslant 1/n\}, \\ C_n &:= \{x \in \mathbb{R}^d : \operatorname{dist}(x, U^c) \leqslant 1/(n+1)\}. \end{aligned}$$

Now $K_n \subset \mathbb{R}^d$ is closed and bounded and therefore compact, and C_n is closed with $K_n \cap C_n = \emptyset$. So by the result given in the hint, there exists a smooth function $\psi_n : \mathbb{R}^d \to \mathbb{R}$ such that $\psi_n \equiv 1$ on K_n and $\psi_n \equiv 0$ on C_n. Define $u_n : \mathbb{R}^d \to \mathbb{R}$ by

$$u_n(x) := u(x)\psi_n(x).$$

Then $u_n \equiv 0$ on $C_n \supseteq U^c$ and $u_n \equiv u$ on $\{z \in U : \operatorname{dist}(z, U^c) \geqslant \frac{1}{n}\}$. Moreover, we have $u_n \in C^2(\mathbb{R}^d; \mathbb{R})$. Now define the process $(Y^n)_{t \geqslant 0}$ by

$$Y_t^n := u_n(X_t^x) \exp\left(\int_0^t c(X_s^x)\,\mathrm{d}s\right).$$

Using Itô's formula, we compute

$$\begin{aligned} Y_t^n = u_n(x) &+ \int_0^t \exp\left(\int_0^s c(X_r^x)\,\mathrm{d}r\right)\Big(Lu_n(X_s^x) + c(X_s^x)u_n(X_s^x)\Big)\,\mathrm{d}s \\ &+ \int_0^t \exp\left(\int_0^s c(X_r^x)\,\mathrm{d}r\right)\nabla u_n(X_s^x)b(X_s^x)\,\mathrm{d}W_s. \end{aligned}$$

As b and c are bounded on $U \subseteq \mathbb{R}$ and u_n has compact support, we can check that the process

$$M_t^n = \int_0^{t\wedge T_n^x} \exp\left(\int_0^s c(X_r^x)\,\mathrm{d}r\right)\nabla u_n(X_s^x)b(X_s^x)\mathrm{d}W_s$$

is in $\mathcal{H}_0^{2,c}$ (since $(X^x)^{T_n^x}$ does not leave U), so that M^n is a true martingale. Taking expectations, we thus obtain

$$E[Y_{t\wedge T_n^x}^n] - u_n(x) = E\Big[\int_0^{t\wedge T_n^x}\Big(Lu_n(X_s^x) + c(X_s^x)u_n(X_s^x)\Big)\exp\left(\int_0^s c(X_r^x)\ \mathrm{d}r\right)\mathrm{d}s\Big].$$

By the definition of T_n^x, we have $u_n(X_{t\wedge T_n^x}^x) = u(X_{t\wedge T_n^x}^x)$ as $\operatorname{dist}(X_{t\wedge T_n^x}^x, U^c) \geqslant \frac{1}{n}$. Moreover, $u_n(x) = u(x)$ since $\operatorname{dist}(x, U^c) > 1/n$. As u solves $(***)$, we get

$$\begin{aligned} u(x) = E&\Big[u(X_{t\wedge T_n^x}^x)\exp\Big(\int_0^{t\wedge T_n^x} c(X_s^x)\,\mathrm{d}s\Big)\Big] \\ &+ E\Big[\int_0^{t\wedge T_n^x} f(X_s^x)\ \exp\Big(\int_0^s c(X_r^x)\ \mathrm{d}r\Big)\mathrm{d}s\Big]. \end{aligned} \tag{1}$$

By continuity of the process $(\mathrm{dist}(X_t^x, U^c))_{t\geqslant 0}$, we have that $T_n^x \uparrow T_U^x < \infty$, which is integrable by assumption. Since $c \leqslant 0$, we have for any $n \in \mathbb{N}$ and $t \geqslant 0$ that

$$\left|u(X^x_{t\wedge T_n^x})\,\exp\Big(\int_0^{t\wedge T_n^x} c(X_s^x)\,\mathrm{d}s\Big)\right| \leqslant \sup_{y\in\overline{U}} |u(y)| < \infty,$$
$$\left|\int_0^{t\wedge T_n^x} f(X_s^x)\exp\Big(\int_0^s c(X_r^x)\,\mathrm{d}r\Big)\right| \leqslant T_U^x \sup_{y\in\overline{U}} |f(y)|.$$

Note that $X^x_{T_U^x} \in \partial U$ by the definition of T_U^x, so that $u(X^x_{T_U^x}) = g(X^x_{T_U^x})$ by $(***)$. Using the dominated convergence theorem, we let $t \to \infty$ and $n \to \infty$ in (1) to conclude that

$$u(x) = E\Big[g(X^x_{T_U^x})\,\exp\Big(\int_0^{T_U^x} c(X_s^x)\,\mathrm{d}s\Big)\Big] + E\Big[\int_0^{T_U^x} f(X_s^x)\,\exp\Big(\int_0^s c(X_r^x)\,\mathrm{d}r\Big)\,\mathrm{d}s\Big],$$

as required.

Exercise 11.3 Consider a probability space $(\Omega, \mathcal{F}, P)$ supporting a Brownian motion $W = (W_t)_{t\geqslant 0}$. Denote by $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$ the P-augmentation of the (raw) filtration generated by W. Let $T > 0$, $\alpha > 0$ and let F be a bounded, $\mathcal{F}_T$-measurable random variable.

(a) Show that the process $X = (X_t)_{0\leqslant t\leqslant T}$ given by

$$X_t = -\alpha \log E[\exp(-F/\alpha)\,|\,\mathcal{F}_t]$$

satisfies the BSDE

$$\mathrm{d}X_t = \frac{1}{2\alpha} Z_t^2\,\mathrm{d}t + Z_t\,\mathrm{d}W_t,$$
$$X_T = F.$$

Hint: We have that $X_t = -\alpha \log Y_t$, where $Y_t := E[\exp(-F/\alpha)\,|\,\mathcal{F}_t]$. Apply Itô's representation theorem to Y_T and Itô's formula to X to derive a solution pair $(X, Z) \in \mathcal{R}^2 \times L^2(W)$ for the BSDE.

(b) Let $b \in \mathbb{R}$. Show that the process $X = (X_t)_{0\leqslant t\leqslant T}$ given by

$$X_t = -\alpha\left(\frac{1}{2}b^2(t-T) - b\,W_t + \log E[\exp(bW_T - F/\alpha)\,|\,\mathcal{F}_t]\right)$$

satisfies the BSDE

$$\mathrm{d}X_t = \left(\frac{1}{2\alpha}\,Z_t^2 - b\,Z_t\right)\mathrm{d}t + Z_t\,\mathrm{d}W_t,$$
$$X_T = F.$$

Solution 11.3

(a) Itô's representation theorem applied to the bounded random variable $\exp(-F/\alpha)$ gives a unique representation

$$\exp(-F/\alpha) = E[\exp(-F/\alpha)] + \int_0^T H_s \, \mathrm{d}W_s$$

for some $H \in L^2_{\mathrm{loc}}(W)$ such that $H \bullet W$ is a true martingale. Since F is bounded, so is $\exp(-F/\alpha)$. Therefore, the continuous martingale $(Y_t)_{0 \leqslant t \leqslant T}$ given by

$$Y_t = \int_0^t H_s \, \mathrm{d}W_s + E[\exp(-F/\alpha)] = E[\exp(-F/\alpha) \,|\, \mathcal{F}_t]$$

is bounded as well. In particular, we have that $(H \bullet W)^T \in \mathcal{H}^{2,c}_0$, so that $H \in L^2(W^T)$. Next, applying Itô's formula to $X_t = -\alpha \log Y_t$ and setting $Z_t := -\frac{\alpha H_t}{Y_t}$ yields

$$\begin{aligned} \mathrm{d}X_t &= -\frac{\alpha}{Y_t} \, \mathrm{d}Y_t + \frac{\alpha}{2Y_t^2} \, \mathrm{d}\langle Y \rangle_t \\ &= -\frac{\alpha H_t}{Y_t} \, \mathrm{d}W_t + \frac{\alpha H_t^2}{2Y_t^2} \, \mathrm{d}t \\ &= Z_t \, \mathrm{d}W_t + \frac{1}{2\alpha} Z_t^2 \, \mathrm{d}t. \end{aligned}$$

So it only remains to show that $(X, Z) \in \mathcal{R}^2 \times L^2(W^T)$. Since F is bounded, we have that $c \leqslant Y \leqslant C$ for some constants $0 < c < C < \infty$. Hence X is also bounded and thus $X \in \mathcal{R}^2$. Since Y is bounded away from 0 in ω and t, we have that $Z \in L^2(W^T)$ as $H \in L^2(W^T)$, as required.

(b) Consider the measure $Q \approx P$ on $\mathcal{F}_T$ with density process

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = e^{bW_t - \frac{1}{2}b^2 t}, \quad 0 \leqslant t \leqslant T.$$

By Girsanov's theorem, we have that $W_t^Q = W_t - bt$ is a Q-Brownian motion on $[0, T]$. Moreover, note that on $[0, T]$, W and W^Q generate the same filtration. We can rewrite the BSDE as

$$\begin{aligned} \mathrm{d}X_t &= \frac{1}{2\alpha} Z_t^2 \, \mathrm{d}t + Z_t \, \mathrm{d}W_t^Q, \\ X_T &= F. \end{aligned}$$

Under Q, the BSDE is as in (a). Thus, we deduce that

$$X_t = -\alpha \log E_Q[\exp(-F/\alpha) \,|\, \mathcal{F}_t]$$

is a solution. Using the definition of Q and Bayes' formula, we obtain that

$$\begin{aligned} X_t &= -\alpha \log E_Q[\exp(-F/\alpha) \mid \mathcal{F}_t] \\ &= -\alpha \log \left(e^{-bW_t + \frac{1}{2}b^2 t} \, E[e^{bW_T - \frac{1}{2}b^2 T} \, \exp(-F/\alpha) \mid \mathcal{F}_t] \right) \\ &= -\alpha \left(\frac{b^2(t-T)}{2} - bW_t + \log E[\exp(bW_T - F/\alpha) \mid \mathcal{F}_t] \right), \end{aligned}$$

completing the proof.