# Brownian Motion and Stochastic Calculus

## Exercise Sheet 12

*Submit by 12:00 on Wednesday, May 21 via the course homepage.*

**Exercise 12.1** Show that any Lévy process $X$ has no fixed time of discontinuity, meaning that $P[\Delta X_t = 0] = 1$ for any $t \geqslant 0$ (where we set $X_{0-} := 0$).

**Exercise 12.2**

(a) Let $N$ be a one-dimensional Poisson process and $Y = (Y_i)_{i\in\mathbb{N}}$ a sequence of i.i.d. $\mathbb{R}^d$-valued random variables which are also independent of $N$. We define the *compound Poisson process* $X = (X_t)_{t\geqslant 0}$ by $X_t := \sum_{j=1}^{N_t} Y_j$. Show that $X$ is a Lévy process and calculate its Lévy triplet.

(b) Does there exist a Lévy process $X$ such that $X_1$ is uniformly distributed on $[0, 1]$?

(c) Let $(X_t)_{t\geqslant 0}$ and $(Y_t)_{t\geqslant 0}$ be $\mathbb{R}^d$-valued processes such that the joint process $(X, Y)$ is Lévy with respect to a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$.

Show that if
$$E[e^{\mathrm{i}u^\top X_t} e^{\mathrm{i}v^\top Y_t}] = E[e^{\mathrm{i}u^\top X_t}]\, E[e^{\mathrm{i}v^\top Y_t}]$$
for all $u, v \in \mathbb{R}^d$ and $t \geqslant 0$, then $X$ and $Y$ are independent.

**Exercise 12.3** Show that any RCLL function $f : \mathbb{R}_+ \to \mathbb{R}$ is bounded and has only countably many jumps on any compact interval.