Brownian Motion and Stochastic Calculus

Exercise Sheet 12

Submit by 12:00 on Wednesday, May 21 via the course homepage.

Exercise 12.1 Show that any Lévy process X has no fixed time of discontinuity, meaning that $P[\Delta X_t = 0] = 1$ for any $t \geqslant 0$ (where we set $X_{0-} := 0$).

Solution 12.1 For $t = 0$, we have $\Delta X_0 = X_0 - X_{0-} = 0$ and so there is nothing to prove. So fix $t > 0$ and $\varepsilon > 0$. Suppose for a fixed $\omega \in \Omega$ that $|\Delta X_t(\omega)| > \varepsilon$. This means that $|\lim_{s\uparrow t} X_s(\omega) - X_t(\omega)| > \varepsilon$, and therefore for all $s < t$ close enough to t, we have $|X_t(\omega) - X_s(\omega)| > \frac{\varepsilon}{2}$. So defining for each $k \in \mathbb{N}$ the set

$$A_k := \left\{ |X_t - X_{t-\frac{1}{k}}| > \frac{\varepsilon}{2} \right\},$$

we have

$$\{|\Delta X_t| > \varepsilon\} \subseteq \bigcup_{n\in\mathbb{N}} \bigcap_{k\geqslant n} A_k = \liminf_{n\to\infty} A_n.$$

Therefore,

$$P[|\Delta X_t| > \varepsilon] = E[\mathbf{1}_{\{|\Delta X_t|>\varepsilon\}}] \leqslant E[\mathbf{1}_{\liminf_{n\to\infty} A_k}] = E[\liminf_{n\to\infty} \mathbf{1}_{A_n}],$$

where the last step uses that $\mathbf{1}_{\liminf_{n\to\infty} B_n} = \liminf_{n\to\infty} \mathbf{1}_{B_n}$ for any sequence of sets $(B_n)_{n\in\mathbb{N}}$. As the $\mathbf{1}_{A_n}$ are nonnegative, we may apply Fatou's lemma to get

$$P[|\Delta X_t| > \varepsilon] \leqslant \liminf_{n\to\infty} E[\mathbf{1}_{A_n}] = \liminf_{n\to\infty} P[A_n] = 0,$$

where the last step uses that X is stochastically continuous and therefore $X_{t-\frac{1}{n}} \to X_t$ in probability as $n \to \infty$. Now as

$$\{\Delta X_t \neq 0\} = \bigcup_{n=1}^{\infty} \left\{ |\Delta X_t| > \frac{1}{n} \right\}$$

is a countable union of nullsets, it follows that $\{\Delta X_t \neq 0\}$ is also a nullset, so that $P[\Delta X_t = 0] = 1$, as required.

Exercise 12.2

(a) Let N be a one-dimensional Poisson process and $Y = (Y_i)_{i\in\mathbb{N}}$ a sequence of i.i.d. $\mathbb{R}^d$-valued random variables which are also independent of N. We define the *compound Poisson process* $X = (X_t)_{t\geqslant 0}$ by $X_t := \sum_{j=1}^{N_t} Y_j$. Show that X is a Lévy process and calculate its Lévy triplet.

(b) Does there exist a Lévy process X such that X_1 is uniformly distributed on $[0, 1]$?

(c) Let $(X_t)_{t\geqslant 0}$ and $(Y_t)_{t\geqslant 0}$ be $\mathbb{R}^d$-valued processes such that the joint process (X, Y) is Lévy with respect to a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$.

Show that if
$$E[e^{\mathrm{i}u^\top X_t} e^{\mathrm{i}v^\top Y_t}] = E[e^{\mathrm{i}u^\top X_t}]\, E[e^{\mathrm{i}v^\top Y_t}]$$
for all $u, v \in \mathbb{R}^d$ and $t \geqslant 0$, then X and Y are independent.

Solution 12.2

(a) Define the discrete-time process $(\widetilde{X}_n)_{n\in\mathbb{N}_0}$ by $\widetilde{X}_n = \sum_{j=1}^{n} Y_j$, with natural filtration given by $\widetilde{\mathcal{F}}_n = \sigma(Y_1, \dots, Y_n)$. It is clear that $\widetilde{X}_0 = 0$ and $\widetilde{X}$ has stationary and independent increments. We also know that the Poisson process N is a Lévy process independent of $\widetilde{X}$. In particular, $\mathcal{F}^N_\infty = \sigma(N_t : t \geqslant 0)$ and $\widetilde{\mathcal{F}}_\infty = \sigma(Y_1, Y_2, \dots)$ are independent σ-fields. We want to show that the process $(X_t)_{t\geqslant 0}$ defined by
$$X_t = \sum_{j=1}^{N_t} Y_j = \widetilde{X}_{N_t}$$
is Lévy. Fix $0 \leqslant t_1 < \cdots < t_m$ and bounded measurable functions f_j. Using the properties of $\widetilde{X}$ and N, we have that
$$\begin{aligned}
E\Bigg[\prod_{j=1}^m f_j(X_{t_j} - X_{t_{j-1}})\Bigg] &= E\Bigg[\prod_{j=1}^m f_j(\widetilde{X}_{N_{t_j}} - \widetilde{X}_{N_{t_{j-1}}})\Bigg] \\
&= E\Bigg[E\Bigg[\prod_{j=1}^m f_j(\widetilde{X}_{N_{t_j}} - \widetilde{X}_{N_{t_{j-1}}}) \,\Bigg|\, \mathcal{F}^N_\infty\Bigg]\Bigg] \\
&= E\Bigg[E\Bigg[\prod_{j=1}^m f_j(\widetilde{X}_{n_j} - \widetilde{X}_{n_{j-1}})\Bigg]\Bigg|_{n_j = N_{t_j}}\Bigg] \\
&= E\Bigg[\Bigg(\prod_{j=1}^m E[f_j(\widetilde{X}_{n_j} - \widetilde{X}_{n_{j-1}})]\Bigg)\Bigg|_{n_j = N_{t_j}}\Bigg] \\
&= E\Bigg[\Bigg(\prod_{j=1}^m E[f_j(\widetilde{X}_{n_j - n_{j-1}})]\Bigg)\Bigg|_{n_j = N_{t_j}}\Bigg].
\end{aligned}$$

Now defining the functions $g_j : \mathbb{N} \to \mathbb{R}$ by $g_j(n) := E[f_j(\widetilde{X}_n)]$, we have

$$\begin{aligned}
E\bigg[\prod_{j=1}^m f_j(X_{t_j} - X_{t_{j-1}})\bigg] &= E\bigg[\prod_{j=1}^m g_j(N_{t_j} - N_{t_{j-1}})\bigg] \\
&= \prod_{j=1}^m E[g_j(N_{t_j} - N_{t_{j-1}})] \\
&= \prod_{j=1}^m E[g_j(N_{t_j - t_{j-1}})] \\
&= \prod_{j=1}^m E\Big[E[f_j(\widetilde{X}_{n_j})]|_{n_j = N_{t_j - t_{j-1}}}\Big] \\
&= \prod_{j=1}^m E\Big[E[f_j(\widetilde{X}_{N_{t_j - t_{j-1}}}) \,|\, \mathcal{F}_\infty^N]\Big] \\
&= \prod_{j=1}^m E[f_j(X_{t_j - t_{j-1}})].
\end{aligned}$$

By the same reasoning as in Exercise 6.3(a), we thus obtain that X is Lévy (since we also have $X_0 = 0$).

Next, we calculate the Lévy triplet. For $u \in \mathbb{R}^d$, we have

$$\begin{aligned}
E[e^{\mathrm{i}u^\top X_t}] &= E\bigg[\sum_{n \geqslant 0} \mathbf{1}_{\{N_t = n\}} \prod_{j=1}^n e^{\mathrm{i}u^\top Y_j}\bigg] = \sum_{n \geqslant 0} P[N_t = n](E[e^{\mathrm{i}u^\top Y_1}])^n \\
&= \sum_{n \geqslant 0} \frac{e^{-\lambda t}(\lambda t)^n}{n!}(E[e^{\mathrm{i}u^\top Y_1}])^n = e^{-\lambda t}\exp(\lambda t E[e^{\mathrm{i}u^\top Y_1}]) \\
&= \exp\Big(\lambda t(E[e^{\mathrm{i}u^\top Y_1}] - 1)\Big).
\end{aligned}$$

Let ν^Y be the distribution of Y_1 and $\nu := \lambda \nu^Y$. Then

$$\lambda(E[e^{\mathrm{i}u^\top Y_1}] - 1) = \lambda \int (e^{\mathrm{i}u^\top x} - 1)\,\mathrm{d}\nu^Y(x) = \int (e^{\mathrm{i}u^\top x} - 1)\,\mathrm{d}\nu(x).$$

Truncating as in the lecture notes, we can decompose

$$\begin{aligned}
E[e^{\mathrm{i}u^\top X_t}] &= \exp\bigg(t\int (e^{\mathrm{i}u^\top x} - 1)\,\mathrm{d}\nu(x)\bigg) \\
&= \exp\bigg(t\bigg(\int iu^\top x \mathbf{1}_{\{|x| \leqslant 1\}}\,\mathrm{d}\nu(x) + \int (e^{\mathrm{i}u^\top x} - 1 - iu^\top x\mathbf{1}_{\{|x| \leqslant 1\}})\,\mathrm{d}\nu(x)\bigg)\bigg).
\end{aligned}$$

Therefore, we obtain the triplet $(b, 0, \nu)$, where $b = \int_{\{x : |x| \leqslant 1\}} x \,\mathrm{d}\nu(x)$.

(b) No. In fact, we can generalise the result to any random variable X_1 with compact support $\mathrm{supp}(X_1) \subseteq [a, b]$, for some $a < b$. To this end, we claim that if X_1 has compact support and is infinitely divisible, then X_1 is constant. This shows that there is no Lévy process X such that X_1 is uniformly distributed on $[0, 1]$.

Now we prove the claim. By infinite divisibility, for each $n \in \mathbb{N}$, we have $X_1 = \sum_{j=1}^{n} Y_j^n$, where the random variables $(Y_j^n)_{j=1,\dots,n}$ are i.i.d. This implies that $\operatorname{supp}(Y_j^n) \subseteq [a/n, b/n]$. Indeed, suppose for contradiction $P[Y_j^n > b/n] > 0$. As the Y_j^n are i.i.d., we then have that

$$P[X_1 > b] \geqslant P\left[\bigcap_{j=1}^{n}\left\{Y_j^n > \frac{b}{n}\right\}\right] = \left(P\left[Y_j^n > \frac{b}{n}\right]\right)^n > 0,$$

which contradicts the fact that $\operatorname{supp}(X_1) \subseteq [a, b]$. The case $P[Y_j^n < a/n] > 0$ is analogous.

Since $\operatorname{supp}(Y_j^n) \subseteq [a/n, b/n]$, we have that $\operatorname{Var}[Y_j^n] \leqslant (b-a)^2/n^2$. Therefore we get $\operatorname{Var}[X_1] \leqslant (b-a)^2/n \to 0$ as $n \to \infty$, so that X_1 is constant, as claimed.

(c) We need to show that $(X_{t_1}, \dots, X_{t_n})$ is independent of $(Y_{t_1}, \dots, Y_{t_n})$ for any $n \in \mathbb{N}$ and $0 = t_0 < t_1 < \dots < t_n$. This follows if we can show that the random variables

$$X_{t_1} - X_{t_0},\ \dots,\ X_{t_n} - X_{t_{n-1}},\ Y_{t_1} - Y_{t_0},\ \dots,\ Y_{t_n} - Y_{t_{n-1}}$$

are independent, because transformations of independent random variables are still independent. For $j = 1, \dots, n$ and $u_j, v_j \in \mathbb{R}^d$, we have that

$$\begin{aligned}
&E\left[\prod_{j=1}^{n} e^{\mathrm{i}u_j^\top (X_{t_j} - X_{t_{j-1}}) + \mathrm{i}v_j^\top (Y_{t_j} - Y_{t_{j-1}})}\right] \\
&= E\left[E\left[\prod_{j=1}^{n} e^{\mathrm{i}u_j^\top (X_{t_j} - X_{t_{j-1}}) + \mathrm{i}v_j^\top (Y_{t_j} - Y_{t_{j-1}})} \,\middle|\, \mathcal{F}_{t_{n-1}}\right]\right] \\
&= E\left[E[e^{\mathrm{i}u_j^\top (X_{t_n} - X_{t_{n-1}}) + \mathrm{i}v_j^\top (Y_{t_n} - Y_{t_{n-1}})} \,|\, \mathcal{F}_{t_{n-1}}] \prod_{j=1}^{n-1} e^{\mathrm{i}u_j^\top (X_{t_j} - X_{t_{j-1}}) + \mathrm{i}v_j^\top (Y_{t_j} - Y_{t_{j-1}})}\right].
\end{aligned}$$

Since (X, Y) is a Lévy process with respect to $\mathbb{F}$, in particular so is $u_j^\top X + v_j^\top Y$, so that $u_j^\top (X_{t_n} - X_{t_{n-1}}) + v_j^\top (Y_{t_n} - Y_{t_{n-1}})$ is independent of $\mathcal{F}_{t_{n-1}}$ and has the same distribution as $u_j^\top X_{t_n - t_{n-1}} + v_j^\top Y_{t_n - t_{n-1}}$. Therefore, the expression above is equal to

$$E[e^{\mathrm{i}u_j^\top X_{t_n - t_{n-1}} + \mathrm{i}v_j^\top Y_{t_n - t_{n-1}}}] E\left[\prod_{j=1}^{n-1} e^{\mathrm{i}u_j^\top (X_{t_j} - X_{t_{j-1}}) + \mathrm{i}v_j^\top (Y_{t_j} - Y_{t_{j-1}})}\right].$$

We can apply an inductive argument to the remaining product to obtain that

$$E\left[\prod_{j=1}^{n} e^{\mathrm{i}u_j^\top (X_{t_j} - X_{t_{j-1}}) + \mathrm{i}v_j^\top (Y_{t_j} - Y_{t_{j-1}})}\right] = \prod_{j=1}^{n} E[e^{\mathrm{i}u_j^\top X_{t_j - t_{j-1}} + \mathrm{i}v_j^\top Y_{t_j - t_{j-1}}}].$$

Finally, by the assumption on X and Y, we have that

$$\begin{aligned}\prod_{j=1}^{n} E[e^{\mathrm{i}u_j^\top X_{t_j-t_{j-1}}+\mathrm{i}v_j^\top Y_{t_j-t_{j-1}}}] &= \prod_{j=1}^{n} E[e^{\mathrm{i}u_j^\top X_{t_j-t_{j-1}}}]E[e^{\mathrm{i}v_j^\top Y_{t_j-t_{j-1}}}] \\ &= \prod_{j=1}^{n} E[e^{\mathrm{i}u_j^\top (X_{t_j}-X_{t_{j-1}})}]E[e^{\mathrm{i}v_j^\top (Y_{t_j}-Y_{t_{j-1}})}].\end{aligned}$$

We have thus shown, as claimed, that the random variables

$$X_{t_1}-X_{t_0},\, \dots,\, X_{t_n}-X_{t_{n-1}},\, Y_{t_1}-Y_{t_0},\, \dots,\, Y_{t_n}-Y_{t_{n-1}}$$

are independent. In particular, the vectors

$$(X_{t_1}-X_{t_0},\, \dots,\, X_{t_n}-X_{t_{n-1}}),\, (Y_{t_1}-Y_{t_0},\, \dots,\, Y_{t_n}-Y_{t_{n-1}})$$

are independent, and therefore so are $(X_{t_1},\dots,X_{t_n})$ and $(Y_{t_1},\dots,Y_{t_n})$. Now since $0=t_0<t_1<\dots<t_n$ were chosen arbitrarily, this shows that X and Y are independent, as required.

Exercise 12.3 Show that any RCLL function $f:\mathbb{R}_+\to\mathbb{R}$ is bounded and has only countably many jumps on any compact interval.

Solution 12.3 For each $\varepsilon>0$, define the set

$$D_\varepsilon := \{t \geqslant 0 : |\Delta f(t)| \geqslant \varepsilon\}$$

of jumps of size at least ε. Fix $0\leqslant a<b$. We claim that $[a,b]\cap D_\varepsilon$ is at most finite, so that f has at most finitely many jumps of size at least ε on the interval $[a,b]$. To this end, suppose for contradiction that $[a,b]\cap D_\varepsilon$ is infinite, and take a countable subset $(t_n)_{n\in\mathbb{N}}\subseteq[a,b]\cap D_\varepsilon$. Then $(t_n)_{n\in\mathbb{N}}$ is a bounded sequence of real numbers, and thus by the Bolzano–Weierstrass theorem, there is a convergent subsequence $(t_{n_k})_{k\in\mathbb{N}}$, say $t_{n_k}\to t_0$ as $k\to\infty$. By taking a further subsequence, we may assume without loss of generality that either $t_{n_k}<t$ for all $k\in\mathbb{N}$ or $t_{n_k}>t$ for all $k\in\mathbb{N}$. We first consider the case that $t_{n_k}<t$ for all $k\in\mathbb{N}$. By the existence of left limits of f, there is some $\delta>0$ such that for all $s_1,s_2\in(t_0-\delta,t_0)$, we have $|f(s_1)-f(s_2)|<\varepsilon$. But also there is some $k\in\mathbb{N}$ with $t_{n_k}\in(t_0-\delta,t_0)$, which would then contradict $t_{n_k}\in D_\varepsilon$. This completes the proof of the first case. Now the other case that $t_{n_k}>t$ for all $k\in\mathbb{N}$ is completely analogous, by the existence of right limits of f.

We have thus shown that $[a,b]\cap D_\varepsilon$ is (at most) finite. Now the set of all discontinuities of f is simply $\bigcup_{n=1}^\infty D_{1/n}$. As a countable union of finite sets is countable, we may conclude that f has only countably many jumps on $[a,b]$.

It remains to show that f is bounded on $[a,b]$. Suppose for contradiction that f is unbounded on $[a,b]$. Then there exists a sequence $(t_n)_{n\in\mathbb{N}}\subseteq[a,b]$ with $f(t_n)\to\infty$ or

$f(t_n) \to -\infty$ as $n \to \infty$. We may assume that $f(t_n) \to \infty$, as the case $f(t_n) \to -\infty$ follows analogously. By the Bolzano–Weierstrass theorem, there exists a convergent subsequence $(t_{n_k})_{k\in\mathbb{N}}$, say $t_{n_k} \to t$ as $k \to \infty$. By considering a further subsequence, we may assume that either $t_{n_k} < t$ for all $k \in \mathbb{N}$ or $t_{n_k} > t$ for all $k \in \mathbb{N}$. We first consider the case where $t_{n_k} < t$ for all $k \in \mathbb{N}$. By the existence of left limits of f, $\lim_{s\uparrow t} f(s) =: L$ exists (in $\mathbb{R}$). But since $f(t_n) \to \infty$, we also must have $f(t_{n_k}) \to \infty$, which contradicts the finiteness of L. As the case where $t_{n_k} > t$ for all $k \in \mathbb{N}$ is analogous, this completes the proof.