Brownian Motion and Stochastic Calculus

Exercise Sheet 13

Submit by 12:00 on Wednesday, May 28 via the course homepage.

Exercise 13.1 Let X be a Lévy process with values in $\mathbb{R}^d$ and $f_t(u) := E[e^{\mathrm{i}u^\top X_t}]$. Recall that X is stochastically continuous, i.e., the map $t \mapsto X_t$ is continuous in probability, and that $f_{t+s}(u) = f_t(u)f_s(u)$ and $f_0(u) = 1$ for all $s, t \geqslant 0$ and $u \in \mathbb{R}^d$.

(a) Show that $f_s(u)^n = f_{ns}(u)$ and $f_t(u) = f_{t/n}(u)^n$ for all $n \in \mathbb{N}$ and $s, t \geqslant 0$.

(b) Show that $t \mapsto f_t(u)$ is right-continuous and $f_t(u) \neq 0$ for all $t \geqslant 0$ and $u \in \mathbb{R}^d$.

(c) By using the central limit theorem, express a standard normal random variable Z as a weak limit of standardised compound Poisson random variables. Make the approximation as explicit as possible in terms of the compound Poisson distributions.

Exercise 13.2 Assume that $X = (X_t)_{t\geqslant 0}$ is a real-valued Lévy process with respect to $\mathbb{F} = (\mathcal{F})_{t\geqslant 0}$ and P and τ is a bounded stopping time. Using only the definition of a Lévy process, show that for all $u \in \mathbb{R}$ and $0 \leqslant s < t$, we have

$$\frac{E[e^{iuX_{\tau+t}}]}{E[e^{iuX_{\tau+s}}]} = E[e^{iuX_{t-s}}].$$

Exercise 13.3

(a) Let $\widetilde{\nu}$ be a finite non-trivial measure supported on $[\varepsilon, \infty)$ for some $\varepsilon > 0$, and set $\widetilde{\lambda} := \widetilde{\nu}([\varepsilon, \infty)) > 0$. Suppose that $(N_t)_{t\geqslant 0}$ is a Poisson process with rate $\widetilde{\lambda}$ and $(\widetilde{Y}_n)_{n\in\mathbb{N}}$ are i.i.d. random variables with distribution $\widetilde{\lambda}^{-1}\widetilde{\nu}$. Check using Exercise 12.2(a) that the process $J_t^{\widetilde{\nu}} := \sum_{j=1}^{N_t} \widetilde{Y}_j$ is a Lévy process with Lévy triplet $(\widetilde{b}, 0, \widetilde{\nu})$, where $\widetilde{b} = \int \mathbf{1}_{[-1,1]}(x)x \,\mathrm{d}\widetilde{\nu}(x)$.

(b) Suppose that $\widetilde{\nu}$ has compact support, i.e., $\widetilde{\nu}((K, \infty)) = 0$ for some $K \in (0, \infty)$, so that $\widetilde{\nu}([\varepsilon, K]) = \widetilde{\lambda}$. Find a constant $\mu > 0$ such that the process $M^{\widetilde{\nu}}$ defined by

$$M_t^{\widetilde{\nu}} := J_t^{\widetilde{\nu}} - \mu t$$

is a martingale. If $\widetilde{\nu}$ is not compactly supported, under what assumption can we find such a constant μ?

(c) For some $K > 0$, let ν be a measure supported on $[0, K]$ such that $\nu(\{0\}) = 0$ and $\nu((\varepsilon, K]) < \infty$ for each $\varepsilon > 0$. Let $(a_m)_{m\in\mathbb{N}_0}$ be a sequence such that $a_0 = K$ and $a_m \downarrow 0$, and let $(\nu_m)_{m\in\mathbb{N}}$ be a sequence of measures that are absolutely continuous with respect to ν with respective densities $\frac{\mathrm{d}\nu_m}{\mathrm{d}\nu} = \mathbf{1}_{(a_m,a_{m-1}]}$. As in part (a), for each $m \in \mathbb{N}$, let $(N_t^m)_{t\geqslant 0}$ be a Poisson process with rate $C_m := \nu((a_m, a_{m-1}])$ and $(Y_n^m)_{n\in\mathbb{N}}$ be i.i.d. random variables with distribution $C_m^{-1}\nu_m$. We suppose that the N^m and Y_n^m are all independent, and define J^{ν_m} and M^{ν_m} as in parts (a) and (b).

Show that for each $k \in \mathbb{N}$, the process $J^k := \sum_{m=1}^k J^{\nu_m}$ is Lévy and find its Lévy triplet. Find a constant μ_k such that $M_t^k := J_t^k - \mu_k t$ is a martingale.

(d) Suppose that $\int_0^K x^2 \,\mathrm{d}\nu(x) < \infty$. For any $T > 0$, show that the sequence of stopped martingales $((M^k)^T)_{k\in\mathbb{N}}$ converges in $\mathcal{H}_0^2$.

(e) Under the assumption in part (d), does $(J^k)_{k\in\mathbb{N}}$ converge?