Brownian Motion and Stochastic Calculus

Exercise Sheet 13

Submit by 12:00 on Wednesday, May 28 via the course homepage.

Exercise 13.1 Let X be a Lévy process with values in $\mathbb{R}^d$ and $f_t(u) := E[e^{\mathrm{i}u^\top X_t}]$. Recall that X is stochastically continuous, i.e., the map $t \mapsto X_t$ is continuous in probability, and that $f_{t+s}(u) = f_t(u) f_s(u)$ and $f_0(u) = 1$ for all $s, t \geqslant 0$ and $u \in \mathbb{R}^d$.

(a) Show that $f_s(u)^n = f_{ns}(u)$ and $f_t(u) = f_{t/n}(u)^n$ for all $n \in \mathbb{N}$ and $s, t \geqslant 0$.

(b) Show that $t \mapsto f_t(u)$ is right-continuous and $f_t(u) \neq 0$ for all $t \geqslant 0$ and $u \in \mathbb{R}^d$.

(c) By using the central limit theorem, express a standard normal random variable Z as a weak limit of standardised compound Poisson random variables. Make the approximation as explicit as possible in terms of the compound Poisson distributions.

Solution 13.1

(a) It follows by induction on n and from $f_{t+s}(u) = f_t(u) f_s(u)$ that $f_s(u)^n = f_{ns}(u)$ for any $s \geqslant 0$ and $n \in \mathbb{N}$. The second claim follows by setting $s = t/n$.

(b) Right-continuity of $t \mapsto f_t(u)$ follows immediately from right-continuity of X and the dominated convergence theorem. Assume that $f_t(u) = 0$ for some $t > 0$ and $u \in \mathbb{R}^d$. Then it follows that $f_{t/n}(u)^n = f_t(u) = 0$, so that $f_{t/n}(u) = 0$ for all $n \in \mathbb{N}$. Taking $n \to \infty$, we obtain a contradiction to the right-continuity at 0 because $f_0(u) = 1$.

(c) Recall that a random variable Y is a compound Poisson random variable if it can be written as

$$Y = \sum_{i=1}^{N} X_i,$$

where N is a Poisson random variable with rate $\lambda > 0$, and $X_1, X_2, \ldots$ are i.i.d. random variables independent of N. Let $E[X_1] := \mu_X$ and $\mathrm{Var}[X_1] := \sigma_X^2$, which we assume are both finite. Now we have

$$E[Y] = E\left[E\left[\sum_{i=1}^{N} X_i \,\middle|\, N\right]\right] = E\Big[N E[X_1]\Big] = E[N]E[X_1] = \lambda \mu_X.$$

Similarly, we have

$$\begin{aligned} E[Y^2] &= E\left[E\left[\sum_{i=1}^N X_i^2 \,\middle|\, N\right]\right] + 2E\left[E\left[\sum_{1\leqslant i<j\leqslant N} X_i X_j \,\middle|\, N\right]\right] \\ &= E[N]E[X_1^2] + E[N(N-1)]E[X_1]^2, \end{aligned}$$

where the last step uses independence of X_i and X_j, and that there are $\binom{N}{2} = \frac{N(N-1)}{2}$ pairs of (i,j) with $1 \leqslant i < j \leqslant N$. Therefore, we have

$$\begin{aligned} \mathrm{Var}[Y] &= E[Y^2] - E[Y]^2 \\ &= E[N]E[X_1^2] + E[N^2]E[X_1]^2 - E[N]E[X_1]^2 - E[N]^2E[X_1]^2 \\ &= E[N]\mathrm{Var}[X_1] + \mathrm{Var}[N]E[X_1]^2 \\ &= \lambda(\mathrm{Var}[X_1] + E[X_1]^2) \\ &= \lambda E[X_1^2]. \end{aligned}$$

Now let $Y_1, Y_2, \ldots$ be i.i.d. copies of Y. For each $n \in \mathbb{N}$, define the sum

$$S_n := \sum_{i=1}^n Y_i.$$

Then

$$E[S_n] = nE[Y] = n\lambda\mu_X$$

and

$$\mathrm{Var}[S_n] = n\mathrm{Var}[Y] = n\lambda E[X_1^2].$$

For the central limit theorem to be non-trivial, we require $\mathrm{Var}[Y] > 0$, so that $\lambda > 0$ and $E[X_1^2] > 0$, i.e. X_1 is not almost surely zero. The central limit theorem then guarantees that, weakly as $n \to \infty$,

$$\frac{S_n - E[S_n]}{\sqrt{\mathrm{Var}[S_n]}} = \frac{S_n - nE[Y]}{\sqrt{n\mathrm{Var}[Y]}} = \frac{S_n - n\lambda\mu_X}{\sqrt{n\lambda\mu_X}} \longrightarrow Z,$$

as required.

Exercise 13.2 Assume that $X = (X_t)_{t\geqslant 0}$ is a real-valued Lévy process with respect to $\mathbb{F} = (\mathcal{F})_{t\geqslant 0}$ and P and τ is a bounded stopping time. Using only the definition of a Lévy process, show that for all $u \in \mathbb{R}$ and $0 \leqslant s < t$, we have

$$\frac{E[e^{iuX_{\tau+t}}]}{E[e^{iuX_{\tau+s}}]} = E[e^{iuX_{t-s}}].$$

Solution 13.2 We can always write $X_{\tau+t} - X_{\tau+s} = X_{\tau+s+(t-s)} - X_{\tau+s}$, and if τ is deterministic, this is independent of $X_{\tau+s}$ and has the same distribution as X_{t-s}.

This also holds for a bounded stopping time τ, but then we use more than the definition of a Lévy process. So we approximate τ and then pass to the limit.

To that end, for each $j, k \in \mathbb{N}$ define the set

$$A_k^{(j)} := \{(k-1)2^{-j} \leqslant \tau < k2^{-j}\} \in \mathcal{F}_{k2^{-j}}.$$

Then define the stopping time

$$\tau_j := \sum_{k=1}^{\infty} k2^{-j} \mathbf{1}_{A_k^{(j)}}.$$

This τ_j is indeed a stopping time since for any $t \geqslant 0$, there is a unique $K \in \mathbb{N}$ with $(K-1)2^{-j} \leqslant t < K2^{-j}$, and so

$$\{\tau_j \leqslant t\} = \bigcup_{k=1}^{K-1} A_k^{(j)} \in \mathcal{F}_{(K-1)2^{-j}} \subseteq \mathcal{F}_t.$$

Moreover, one can see that $\tau_j \downarrow \tau$ P-a.s. as $j \to \infty$. Now fix $0 \leqslant s < t$. Since $\tau_j \downarrow \tau$ P-a.s. as $j \to \infty$, we have by the right-continuity of X that $e^{iuX_{\tau_j+t}} \to e^{iuX_{\tau+t}}$ P-a.s. as $j \to \infty$. As these random variables are bounded by 1, we may apply the dominated convergence theorem to get for any $u \in \mathbb{R}$ that

$$E[e^{iuX_{\tau+t}}] = \lim_{j\to\infty} E[e^{iuX_{\tau_j+t}}] = \lim_{j\to\infty} \sum_{k=1}^{\infty} E[e^{iuX_{k2^{-j}+t}} \mathbf{1}_{A_k^{(j)}}].$$

Now for each $j, k \in \mathbb{N}$, we write

$$\begin{aligned} E[e^{iuX_{k2^{-j}+t}} \mathbf{1}_{A_k^{(j)}}] &= E[e^{iu(X_{k2^{-j}+t} - X_{k2^{-j}+s})} e^{iuX_{k2^{-j}+s}} \mathbf{1}_{A_k^{(j)}}] \\ &= E[e^{iu(X_{k2^{-j}+t} - X_{k2^{-j}+s})}] E[e^{iuX_{k2^{-j}+s}} \mathbf{1}_{A_k^{(j)}}], \end{aligned}$$

where the last step uses that the increment $X_{k2^{-j}+t} - X_{k2^{-j}+s}$ is independent of $\mathcal{F}_{k2^{-j}+s}$ and $e^{iuX_{k2^{-j}+s}} \mathbf{1}_{A_k^{(j)}}$ is $\mathcal{F}_{k2^{-j}+s}$-measurable. As X has stationary increments, we have

$$E[e^{iu(X_{k2^{-j}+t} - X_{k2^{-j}+s})}] = E[e^{iuX_{t-s}}].$$

So putting the pieces together, we get

$$\begin{aligned} E[e^{iuX_{\tau+t}}] &= E[e^{iuX_{t-s}}] \lim_{j\to\infty} \sum_{k=1}^{\infty} E[e^{iuX_{k2^{-j}+s}} \mathbf{1}_{A_k^{(j)}}] \\ &= E[e^{iuX_{t-s}}] \lim_{j\to\infty} E[e^{iuX_{\tau_j+s}}] \\ &= E[e^{iuX_{t-s}}] E[e^{iuX_{\tau+s}}], \end{aligned}$$

where the last step again uses the dominated convergence theorem. This completes the proof.

Exercise 13.3

(a) Let $\widetilde{\nu}$ be a finite non-trivial measure supported on $[\varepsilon, \infty)$ for some $\varepsilon > 0$, and set $\widetilde{\lambda} := \widetilde{\nu}([\varepsilon, \infty)) > 0$. Suppose that $(N_t)_{t\geqslant 0}$ is a Poisson process with rate $\widetilde{\lambda}$ and $(\widetilde{Y}_n)_{n\in\mathbb{N}}$ are i.i.d. random variables with distribution $\widetilde{\lambda}^{-1}\widetilde{\nu}$. Check using Exercise 12.2(a) that the process $J_t^{\widetilde{\nu}} := \sum_{j=1}^{N_t} \widetilde{Y}_j$ is a Lévy process with Lévy triplet $(\widetilde{b}, 0, \widetilde{\nu})$, where $\widetilde{b} = \int \mathbf{1}_{[-1,1]}(x)x \, \mathrm{d}\widetilde{\nu}(x)$.

(b) Suppose that $\widetilde{\nu}$ has compact support, i.e., $\widetilde{\nu}((K, \infty)) = 0$ for some $K \in (0, \infty)$, so that $\widetilde{\nu}([\varepsilon, K]) = \widetilde{\lambda}$. Find a constant $\mu > 0$ such that the process $M^{\widetilde{\nu}}$ defined by

$$M_t^{\widetilde{\nu}} := J_t^{\widetilde{\nu}} - \mu t$$

is a martingale. If $\widetilde{\nu}$ is not compactly supported, under what assumption can we find such a constant μ?

(c) For some $K > 0$, let ν be a measure supported on $[0, K]$ such that $\nu(\{0\}) = 0$ and $\nu((\varepsilon, K]) < \infty$ for each $\varepsilon > 0$. Let $(a_m)_{m\in\mathbb{N}_0}$ be a sequence such that $a_0 = K$ and $a_m \downarrow 0$, and let $(\nu_m)_{m\in\mathbb{N}}$ be a sequence of measures that are absolutely continuous with respect to ν with respective densities $\frac{\mathrm{d}\nu_m}{\mathrm{d}\nu} = \mathbf{1}_{(a_m, a_{m-1}]}$. As in part (a), for each $m \in \mathbb{N}$, let $(N_t^m)_{t\geqslant 0}$ be a Poisson process with rate $C_m := \nu((a_m, a_{m-1}])$ and $(Y_n^m)_{n\in\mathbb{N}}$ be i.i.d. random variables with distribution $C_m^{-1}\nu_m$. We suppose that the N^m and Y_n^m are all independent, and define J^{ν_m} and M^{ν_m} as in parts (a) and (b).

Show that for each $k \in \mathbb{N}$, the process $J^k := \sum_{m=1}^k J^{\nu_m}$ is Lévy and find its Lévy triplet. Find a constant μ_k such that $M_t^k := J_t^k - \mu_k t$ is a martingale.

(d) Suppose that $\int_0^K x^2 \, \mathrm{d}\nu(x) < \infty$. For any $T > 0$, show that the sequence of stopped martingales $((M^k)^T)_{k\in\mathbb{N}}$ converges in $\mathcal{H}_0^2$.

(e) Under the assumption in part (d), does $(J^k)_{k\in\mathbb{N}}$ converge?

Solution 13.3

(a) This is an immediate check from Exercise 12.2(a). Note that $J^{\widetilde{\nu}}$ is a compound Poisson process, where the jumps $\widetilde{Y}_n$ have distribution $F = \widetilde{\lambda}^{-1}\widetilde{\nu}$. Therefore, $J^{\widetilde{\nu}}$ is a Lévy process with triplet $(\widetilde{b}, 0, \widetilde{\lambda}\widetilde{\lambda}^{-1}\widetilde{\nu}) = (\widetilde{b}, 0, \widetilde{\nu})$, as required.

(b) Note that $(J_t^{\widetilde{\nu}} - \mu t)_{t\geqslant 0}$ is a Lévy process for any $\mu > 0$. Moreover, $J_1^{\widetilde{\nu}} \in L^1$ because $J^{\widetilde{\nu}}$ is a compound Poisson process and $\widetilde{Y}_1 \in L^1$, since it has compact support. Therefore, it is enough to check that $E[J_1^{\widetilde{\nu}} - \mu] = 0$. Indeed, we have by independence that

$$\begin{aligned} E[J_1^{\widetilde{\nu}}] &= E\left[\sum_{n=1}^{N_1} Y_n\right] = E\left[E\left[\sum_{n=1}^{\widehat{n}} Y_n\right]\bigg|_{\widehat{n}=N_1}\right] \\ &= E\left[N_1\widetilde{\lambda}^{-1}\int_\varepsilon^K x \, \mathrm{d}\widetilde{\nu}(x)\right] = \int_\varepsilon^K x \, \mathrm{d}\widetilde{\nu}(x). \end{aligned}$$

For $\mu = \int_\varepsilon^K x \, \mathrm{d}\widetilde{\nu}(x)$, we thus have that $J_1^{\widetilde{\nu}} - \mu$ is integrable with $E[J_1^{\widetilde{\nu}} - \mu] = 0$. Therefore, $M^{\widetilde{\nu}}$ is a martingale by Proposition 5.2.2 of the notes. If $\widetilde{\nu}$ is not compactly supported, we can still find $\mu = \int_\varepsilon^\infty x \, \mathrm{d}\widetilde{\nu}(x)$ by the same argument, as long as the integral is finite.

(c) It follows immediately from part (a) that each J^{ν_m} is a Lévy process with triplet $(b_m, 0, \nu_m)$. Moreover, the $(J^{\nu_m})_{m\in\mathbb{N}}$ are independent by construction. One easily checks that the sum of independent Lévy processes $J^k := \sum_{m=1}^k J^{\nu_m}$ is also Lévy. Moreover, J^k has the Lévy triplet $(\widehat{b}_k, 0, \sum_{m=1}^k \nu_m) = (\widehat{b}_k, 0, \widehat{\nu}_k)$, where $\widehat{\nu}_k$ has density $\frac{\mathrm{d}\widehat{\nu}_k}{\mathrm{d}\nu} = \mathbf{1}_{(a_k,K]}$ and $\widehat{b}_k = \int \mathbf{1}_{[-1,1]}(x) x \, \mathrm{d}\widehat{\nu}_k(x)$. As in (b), we have that μ_k is given by

$$\mu_k = \int x \, \mathrm{d}\widehat{\nu}_k(x) = \int \mathbf{1}_{(a_m,K]}(x) x \, \mathrm{d}\nu(x).$$

(d) We show that $(M^k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}_0^2$. Note first that we have $\|M^T\|_{\mathcal{H}_0^2} = E[[M]_T]$ and each M^k has finite variation. Also, by independence, the probability of two of the processes $J^{\nu_m}, J^{\nu_{m'}}$ jumping simultaneously is 0. Therefore, for $k' \geqslant k$, we can compute

$$\begin{aligned}
E\Big[[M^{k'} - M^k]_T\Big] &= E\bigg[\sum_{0<s\leqslant T} \Delta(M^{k'} - M^k)_s^2\bigg] \\
&= E\bigg[\sum_{0<s\leqslant T} \sum_{m=k+1}^{k'} (\Delta J_s^{\nu_m})^2\bigg] \\
&= E\bigg[\sum_{m=k+1}^{k'} \sum_{n=1}^{N_T^m} (Y_n^m)^2\bigg] \\
&= \sum_{m=k+1}^{k'} E\bigg[E\bigg[\sum_{n=1}^{\widehat{n}} (Y_n^m)^2\bigg]\bigg|_{\widehat{n}=N_T^m}\bigg] \\
&= \sum_{m=k+1}^{k'} E\bigg[N_T^m (\lambda_m)^{-1} \int \mathbf{1}_{(a_m,a_{m-1}]}(x) x^2 \, \mathrm{d}\nu(x)\bigg] \\
&= \sum_{m=k+1}^{k'} \int \mathbf{1}_{(a_m,a_{m-1}]}(x) x^2 \, \mathrm{d}\nu(x) \\
&= \int \mathbf{1}_{(a_{k'},a_k]}(x) x^2 \, \mathrm{d}\nu(x) \\
&\leqslant \int \mathbf{1}_{[0,a_k]}(x) x^2 \, \mathrm{d}\nu(x).
\end{aligned}$$

Since we assume that $\int_0^K x^2 \, \mathrm{d}\nu(x) < \infty$, it follows from the dominated convergence theorem that $\lim_{k\to\infty} \sup_{k'\geqslant k} E[[M^{k'} - M^k]_T] = 0$. Thus $((M^k)^T)_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}_0^2$ and is therefore convergent.

(e) No, $(J^k)_{k\in\mathbb{N}}$ is not convergent in general. As a counterexample, we let ν have density with respect to Lebesgue measure $\frac{\mathrm{d}\nu(x)}{\mathrm{d}x} = \mathbf{1}_{(0,1]}(x) x^{-2}$. Setting

$a_m = e^{-m}$, we have that

$$\mu_k = \int_{e^{-k}}^{1} x^{-1}\,\mathrm{d}x = [\log x]_{e^{-k}}^{1} = k \to \infty.$$

As $((M^k)^1)_{k\in\mathbb{N}}$ converges in $\mathcal{H}_0^2$, in particular (M_1^k) converges in probability. But since $(\mu_k)_{k\in\mathbb{N}}$ diverges, it follows that $(J^k)_{k\in\mathbb{N}}$ cannot converge, as claimed.