Coordinator
Benjamin Kotlov

Brownian Motion and Stochastic Calculus

Exercise Sheet 1

Submit by 12:00 on Wednesday, February 26 via the course homepage.

Exercise 1.1 *(Monotone class theorem)* Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X, Y : \Omega \to \mathbb{R}$ random variables. Use the monotone class theorem to show that a random variable Z is $\sigma(X,Y)$-measurable if and only if it is of the form $Z = f(X,Y)$ for some (Borel)-measurable function $f : \mathbb{R}^2 \to \mathbb{R}$.

Hint: It may be helpful to start by assuming that Z is bounded.

Solution 1.1 Since the random variable $Z = f(X,Y)$ is $\sigma(X,Y)$-measurable for any measurable $f : \mathbb{R}^2 \to \mathbb{R}$, it only remains to prove the converse.

Define the sets $\mathcal{H}$ and $\mathcal{M}$ by

$$\begin{aligned}\mathcal{H} &:= \{Z = f(X,Y) : Z \text{ bounded}, f : \mathbb{R}^2 \to \mathbb{R} \text{ measurable}\},\\ \mathcal{M} &:= \{\mathbf{1}_{\{X\in A\}}\mathbf{1}_{\{Y\in B\}} : A, B \in \mathcal{B}(\mathbb{R})\}.\end{aligned}$$

We check that $\mathcal{H}$ and $\mathcal{M}$ satisfy the conditions of the monotone class theorem. It is clear that $\mathcal{M}$ is closed under multiplication, and that $\mathcal{H}$ is a real vector space of bounded functions that contains $\mathcal{M}$ and the constant function 1. It remains to check that $\mathcal{H}$ is closed under monotone bounded convergence. To this end, suppose that $(Z_k)_{k\in\mathbb{N}}$ is a nondecreasing sequence of nonnegative random variables where each Z_k is of the form $Z_k = f_k(X,Y)$ for some measurable $f_k : \mathbb{R}^2 \to \mathbb{R}$. Suppose moreover that $Z := \sup_{k\in\mathbb{N}} Z_k = \lim_{k\to\infty} Z_k$ is bounded. Let $C > 0$ be an upper bound on Z, and define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) := \sup_{k\in\mathbb{N}} f_k(x,y) \wedge C.$$

Then f is measurable and $Z = f(X,Y)$, so that $Z \in \mathcal{H}$. As $\sigma(\mathcal{M}) = \sigma(X,Y)$, the monotone class theorem asserts that $\mathcal{H}$ contains all bounded $\sigma(X,Y)$-measurable random variables. It now only remains to show that any (not necessarily bounded) $\sigma(X,Y)$-measurable random variable is of the form $f(X,Y)$ for some measurable $f : \mathbb{R}^2 \to \mathbb{R}$.

So consider a $\sigma(X,Y)$-measurable random variable Z. By writing $Z = Z^+ - Z^-$ and arguing for Z^+ (and analogously for Z^-), we may assume without loss of generality that $Z \geqslant 0$. For each $n \in \mathbb{N}$, let $Z^n := Z \wedge n$. Then Z^n is $\sigma(X,Y)$-measurable and bounded random variable, and so there exists measurable $f^n : \mathbb{R}^2 \to \mathbb{R}$ such that $Z^n = f^n(X, Y)$. By construction, we have $Z^n \to Z$ pointwise. Moreover, the sequence $(Z^n)_{n\in\mathbb{N}}$ is increasing, so that $f^n(X, Y) \leqslant f^{n+1}(X, Y)$ for each $n \in \mathbb{N}$. Thus, by defining $h^{n+1} := f^n \vee f^{n+1}$, we have $h^{n+1}(X, Y) = f^{n+1}(X, Y)$. Now $(h^n)_{n\in\mathbb{N}}$ is an increasing sequence of functions, and thus $h := \lim_{n\to\infty} h^n = \sup_{n\in\mathbb{N}} h^n$ exists. As the h^n are measurable, so is h, and we have

$$h(X, Y) = \lim_{n\to\infty} h^n(X, Y) = \lim_{n\to\infty} f^n(X, Y) = \lim_{n\to\infty} Z^n = Z.$$

This completes the proof.

Exercise 1.2 *(Modifications and indistinguishability)* Let $(\Omega, \mathcal{F}, P)$ be a probability space and assume that $X = (X_t)_{t\geqslant 0}$ and $Y = (Y_t)_{t\geqslant 0}$ are two stochastic processes on $(\Omega, \mathcal{F}, P)$. Recall that two processes Z and Z' on $(\Omega, \mathcal{F}, P)$ are said to be *modifications* of each other if $P[Z_t = Z'_t] = 1$ for each $t \geqslant 0$, while Z and Z' are *indistinguishable* if $P[Z_t = Z'_t \text{ for all } t \geqslant 0] = 1$.

(a) Assume that X and Y are both P-a.s. right-continuous or both P-a.s. left-continuous. Show that the processes are modifications of each other if and only if they are indistinguishable.

(b) Give an example showing that one of the implications in part (a) does not hold for general stochastic processes X and Y.

Solution 1.2

(a) For a fixed $t_0 \geqslant 0$, we have

$$\{X_{t_0} = Y_{t_0}\} \supseteq \{X_t = Y_t \text{ for all } t \geqslant 0\}.$$

So if X and Y are indistinguishable, it follows that $P[X_{t_0} = Y_{t_0}] = 1$, so that X and Y are modifications of each other.

Conversely, suppose that X is a modification of Y. We need to show that if X and Y are both right-continuous, then they are indistinguishable (the proof for the left-continuous case is analogous). Consider the set

$$\begin{aligned} A &:= \bigcap_{t\in\mathbb{Q}_+} \{\omega \in \Omega : X_t(\omega) = Y_t(\omega)\} \\ &= \{\omega \in \Omega : X_t(\omega) = Y_t(\omega) \text{ for all } t \in \mathbb{Q}_+\}. \end{aligned}$$

Since X and Y are modification of each other and the above intersection is countable, it follows that $P[A] = 1$ (as the countable intersection of probability-1 events is a probability-1 event). Let B be a probability-1 event on which X and Y are both right-continuous, and define the probability-1 event C by

$$\begin{aligned} C &:= A \cap B \\ &= \{\omega \in \Omega : X_t(\omega) = Y_t(\omega) \text{ for all } t \in \mathbb{Q}_+, X_\cdot(\omega) \text{ and } Y_\cdot(\omega) \text{ right-cts}\}. \end{aligned}$$

Now fix $\omega \in C$ and take some $t_0 \geqslant 0$. Since $X_\cdot(\omega)$ and $Y_\cdot(\omega)$ are right-continuous, then for some (any) decreasing sequence of rationals $(t_n)_{n\in\mathbb{N}}$ with $t_n \downarrow t_0$, we have

$$X_{t_0}(\omega) = \lim_{n\to\infty} X_{t_n}(\omega) = \lim_{n\to\infty} Y_{t_n}(\omega) = Y_{t_0}(\omega),$$

where the second equality above follows from the fact that $X_t(\omega) = Y_t(\omega)$ for all $t \in \mathbb{Q}_+$. We have thus shown that

$$\{\omega \in \Omega : X_t(\omega) = Y_t(\omega) \text{ for all } t \geqslant 0\} \supseteq C,$$

implying that $P[X_t = Y_t \text{ for all } t \geqslant 0] = 1$. (To be precise, we have shown that the complement of the set $\{X_t = Y_t \text{ for all } t \geqslant 0\}$ is contained in a P-nullset and thus has outer P-measure 0.) This completes the proof.

(b) Take $\Omega = [0,\infty)$, $\mathcal{F} = \mathcal{B}([0,\infty))$ the Borel σ-algebra, and P a probability measure with $P[\{\omega\}] = 0,\ \forall\, \omega \in \Omega$ (for instance, the exponential distribution).

Set $X \equiv 0$ and

$$Y_t(\omega) = \begin{cases} 1, & \omega = t, \\ 0, & \text{otherwise.} \end{cases}$$

Then for each $t \geqslant 0$, we have

$$P[X_t \neq Y_t] = P[Y_t = 1] = P[\{t\}] = 0,$$

so that X and Y are modifications of each other. However,

$$P[X_t = Y_t \text{ for all } t \geqslant 0] = P[Y_t = 0 \text{ for all } t \geqslant 0] = P[\emptyset] = 0,$$

and thus X and Y are not indistinguishable. Note here that Y is neither right-continuous nor left-continuous, as we should expect.

Exercise 1.3 *(Measurability of stochastic processes)* Let $X = (X_t)_{t\geqslant 0}$ be a stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, P)$. The aim of this exercise is to show the following chain of implications:

X optional $\Rightarrow$ X progressively measurable $\Rightarrow$ X product-measurable and adapted.

(a) Show that every progressively measurable process is product-measurable and adapted.

(b) Assume that X is adapted and *every* path of X is right-continuous. Show that X is progressively measurable.

Remark: The same conclusion holds true if every path of X is left-continuous.

Hint: For fixed $t \geqslant 0$, consider the approximating sequence of processes Y^n on $\Omega \times [0,t]$ given by $Y^n_0 = X_0$ and $Y^n_u = \sum_{k=0}^{2^n-1} \mathbf{1}_{(tk2^{-n}, t(k+1)2^{-n}]}(u) X_{t(k+1)2^{-n}}$ for $u \in (0,t]$.

(c) Recall that the optional σ-field $\mathcal{O}$ is generated by the class $\overline{\mathcal{M}}$ of all adapted processes whose paths are all RCLL. Show that $\mathcal{O}$ is also generated by the subclass $\mathcal{M}$ of all *bounded* processes in $\overline{\mathcal{M}}$.

(d) Use the monotone class theorem to show that every optional process is progressively measurable.

Solution 1.3

(a) Suppose that X is progressively measurable, so that for each $t \geqslant 0$, $X|_{\Omega\times[0,t]}$ is $\mathcal{F}_t \otimes \mathcal{B}[0,t]$-measurable. For fixed $t \geqslant 0$, the map

$$\begin{aligned} i_t : (\Omega, \mathcal{F}_t) &\to (\Omega \times [0,t], \mathcal{F}_t \otimes \mathcal{B}[0,t]), \\ \omega &\mapsto (\omega, t) \end{aligned}$$

is measurable, and thus $X_t = X \circ i_t$ is $\mathcal{F}_t$-measurable. Moreover, the processes X^n defined by $X^n_t := X|_{\Omega\times[0,n]}\mathbf{1}_{[0,n]}(t)$, $t \geqslant 0$, are $\mathcal{F}\otimes\mathcal{B}[0,\infty)$-measurable. Since $X^n \to X$ pointwise (in (t,ω)) as $n \to \infty$, also X is $\mathcal{F} \otimes \mathcal{B}[0,\infty)$-measurable.

(b) Fix $t \geqslant 0$ and consider the sequence of processes Y^n on $\Omega \times [0,t]$ as in the hint. By construction, each Y^n is $\mathcal{F}_t \otimes \mathcal{B}[0,t]$-measurable. Since X is right-continuous, we have $Y^n \to X|_{\Omega\times[0,t]}$ pointwise as $n \to \infty$, and thus $X|_{\Omega\times[0,t]}$ is also $\mathcal{F}_t \otimes \mathcal{B}[0,t]$-measurable, as required.

(c) Let X be adapted with all paths being RCLL. For each $n \in \mathbb{N}$, consider the processes $X^n := (X \wedge n) \vee (-n)$. Each X^n is bounded and RCLL, and thus $\sigma(\mathcal{M})$-measurable. But as $X^n \to X$ pointwise (in $((t,\omega))$, X is also $\sigma(\mathcal{M})$-measurable. It follows immediately that $\mathcal{O} \subseteq \sigma(\mathcal{M})$, and as $\sigma(\mathcal{M}) \subseteq \mathcal{O}$, this completes the proof.

(d) It suffices to show that all bounded optional processes are progressively measurable, since for a general optional process X, we then have that $X^n := X\mathbf{1}_{\{|X|\leqslant n\}}$, a bounded optional process, is progressively measurable, which then implies that X is also progressively measurable because $X^n \to X$ pointwise.

So let $\mathcal{H}$ denote the real vector space of bounded and progressively measurable processes. By part (b), we have $\mathcal{H} \supseteq \mathcal{M}$, and of course $\mathcal{H}$ contains the constant process 1 and is closed under monotone bounded convergence. Also, $\mathcal{M}$ is closed under multiplication. The monotone class theorem then yields that every bounded $\sigma(\mathcal{M})$-measurable process is contained in $\mathcal{H}$. By part (c), we can then conclude that every bounded optional process is progressively measurable. This completes the proof.

Exercise 1.4 *(Transformations of Brownian motion)* Let $W = (W_t)_{t\geqslant 0}$ be a Brownian motion.

(a) Show that $(-W_t)_{t\geqslant 0}$ is a Brownian motion.

(b) Show that for any $c \neq 0$, $(cW_{t/c^2})_{t\geqslant 0}$ is also a Brownian motion.

Solution 1.4

(a) We have $P[-W_0 = 0] = P[W_0 = 0] = 1$, so that (BM1) holds. Next, for any fixed $n \in \mathbb{N}$ and $0 \leqslant t_0 < t_1 < \cdots < t_n$, the increments $W_{t_i} - W_{t_{i-1}}$, $i = 1, \ldots, n$, are independent and distributed $\mathcal{N}(0, t_i - t_{i-1})$. We then have that the increments

$$-W_{t_i} - (-W_{t_{i-1}}) = -(W_{t_i} - W_{t_{i-1}})$$

are independent (as the map $x \mapsto -x$ is measurable on $\mathbb{R}$, and independence is preserved under measurable transformations) and $\sim \mathcal{N}(0, t_i - t_{i-1})$. Thus (BM2$'$) holds. Lastly, P-almost all trajectories $t \mapsto -W_t(\omega)$ are continuous, since for any $\omega \in \Omega$ with $t \mapsto W_t(\omega)$ being continuous, we also have that $t \mapsto -W_t(\omega)$ is continuous, and hence (BM3) holds. We thus have that $-W$ is a Brownian motion.

(b) We have $P[cW_{0/c^2} = 0] = P[cW_0 = 0] = P[W_0 = 0] = 1$, so that (BM1) holds. Next fix $n \in \mathbb{N}$ and $0 \leqslant t_0 < t_1 < \cdots < t_n$. We can write

$$cW_{t_i/c^2} - cW_{t_{i-1}/c^2} = c(W_{t_i/c^2} - W_{t_{i-1}/c^2}).$$

We know that $W_{t_i/c^2} - W_{t_{i-1}/c^2} \sim \mathcal{N}(0, t_i/c^2 - t_{i-1}/c^2)$, and thus we have $cW_{t_i/c^2} - cW_{t_{i-1}/c^2} \sim \mathcal{N}(0, t_i - t_{i-1})$. Moreover, the increments $W_{t_i/c^2} - W_{t_{i-1}/c^2}$, $i = 1, \ldots, n$, are independent (as $0 \leqslant t_0/c^2 < t_1/c^2 < \cdots < t_n/c^2$), and thus so are $cW_{t_i/c^2} - cW_{t_{i-1}/c^2}$, $i = 1, \ldots, n$, since the map $x \mapsto cx$ is measurable on $\mathbb{R}$. Thus (BM2$'$) holds. Lastly, for each ω with $t \mapsto W_t(\omega)$ continuous, we have that $t \mapsto cW_{t/c^2}(\omega)$ is also continuous, as it is a composition of continuous functions. Thus (BM3) holds, completing the proof.