Coordinator
Benjamin Kotlov

Brownian Motion and Stochastic Calculus

Exercise Sheet 2

Submit by 12:00 on Wednesday, March 5 via the course homepage.

Exercise 2.1 *(Equivalent definitions of Brownian motion)* Let X be a stochastic process on a probability space $(\Omega, \mathcal{F}, P)$ with $X_0 = 0$ P-a.s., and let $\mathbb{F}^X = (\mathcal{F}_t^X)_{t \geqslant 0}$ denote the (raw) filtration generated by X, i.e., $\mathcal{F}_t^X = \sigma(X_s;\, s \leqslant t)$. Show that the following two properties are equivalent:

(i) X has *independent increments*, i.e., for all $n \in \mathbb{N}$ and $0 \leqslant t_0 < t_1 < \cdots < t_n$, the increments $X_{t_i} - X_{t_{i-1}}$, $i = 1, \ldots, n$, are independent.

(ii) X has $\mathbb{F}^X$*-independent increments*, i.e., $X_t - X_s$ is independent of $\mathcal{F}_s^X$ whenever $t \geqslant s$.

Remark: This shows the equivalence of the properties (BM2) and (BM2′) of Brownian motion.

Hint: For proving "(i) ⇒ (ii)", you might use the monotone class theorem. When choosing the set $\mathcal{H}$, recall that a random variable Y is independent of a σ-algebra $\mathcal{G}$ if and only if $E[g(Y)Z] = E[g(Y)]E[Z]$ for all bounded measurable functions $g : \mathbb{R} \to \mathbb{R}$ and bounded $\mathcal{G}$-measurable random variables Z.

Exercise 2.2 *(Hölder continuity of Brownian paths)* For a fixed $\alpha > 0$, a function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is called *locally α-Hölder-continuous at a point $x \in D$* if there exist $\delta > 0$ and $C > 0$ such that $|f(x) - f(y)| \leqslant C|x - y|^\alpha$ for all $y \in D$ with $|x - y| \leqslant \delta$. If f is locally α-Hölder-continuous at every $x \in D$, we say that f is *locally α-Hölder-continuous.*

(a) Let $Z \sim \mathcal{N}(0,1)$. Show that $P[|Z| \leqslant \varepsilon] \leqslant \varepsilon$ for any $\varepsilon \geqslant 0$.

(b) Let W be a Brownian motion. Prove that for any $\alpha > \frac{1}{2}$, P-almost all paths of W are nowhere locally α-Hölder-continuous on $[0,1]$.

Hint: Take any $M \in \mathbb{N}$ satisfying $M(\alpha - \frac{1}{2}) > 1$ and show that the set $\{W_\cdot(\omega) \text{ is locally } \alpha\text{-Hölder at some } s \in [0,1]\}$ is contained in the set $\bigcup_{C \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \bigcap_{n \geqslant m} \bigcup_{k=0,\ldots,n-M} \bigcap_{j=1}^{M} \{|W_{\frac{k+j}{n}}(\omega) - W_{\frac{k+j-1}{n}}(\omega)| \leqslant C \frac{1}{n^\alpha}\}$.

(c) The *Kolmogorov–Čentsov theorem* states that a stochastic process X on $[0,T]$ satisfying

$$E[|X_t - X_s|^\gamma] \leqslant C\,|t - s|^{1+\beta}, \quad s, t \in [0,T],$$

for some fixed $\gamma, \beta, C > 0$ has a version which is locally α-Hölder-continuous for each $\alpha < \beta/\gamma$. Use this result to deduce that Brownian motion is P-a.s. locally α-Hölder-continuous for every $\alpha < 1/2$.

Remark: One can also show that the Brownian paths are *not* locally 1/2-Hölder-continuous. The exact modulus of continuity was found by P. Lévy.

Exercise 2.3 *(A new Brownian motion)* Let $(\Omega, \mathcal{F}, P)$ be a probability space, $W = (W_t)_{t\geqslant 0}$ a Brownian motion on $(\Omega, \mathcal{F}, P)$, Z a random variable independent of W and $s \in (0, \infty)$ a fixed time. We define the stochastic process $V = (V_t)_{t\geqslant 0}$ by

$$V_t := W_t \mathbf{1}_{\{t<s\}} + \Big(W_s + Z(W_t - W_s)\Big)\mathbf{1}_{\{t\geqslant s\}}.$$

Find all possible distributions of Z such that V is a Brownian motion.

Exercise 2.4 *(Blumenthal's 0-1 law)*

(a) Let W be a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ with natural filtration $(\mathcal{F}_t)_{t\geqslant 0}$, i.e. $\mathcal{F}_t = \sigma(W_s, 0 \leqslant s \leqslant t)$. Consider the σ-field

$$\mathcal{F}_{0+} := \bigcap_{t>0} \mathcal{F}_t.$$

Establish *Blumenthal's 0-1 law*: for $A \in \mathcal{F}_{0+}$, either $P[A] = 0$ or $P[A] = 1$.

(b) Show that

$$P\left[\limsup_{t\downarrow 0} \frac{W_t}{\sqrt{t}} = \infty\right] = 1.$$

Hint: Start by showing that for each $C > 0$,

$$\lim_{t\downarrow 0} P\left[\sup_{0\leqslant s\leqslant t} (W_s - C\sqrt{s}) > 0\right] > 0$$

and then use part (a).