Brownian Motion and Stochastic Calculus

Exercise Sheet 2

Submit by 12:00 on Wednesday, March 5 via the course homepage.

Exercise 2.1 *(Equivalent definitions of Brownian motion)* Let X be a stochastic process on a probability space $(\Omega, \mathcal{F}, P)$ with $X_0 = 0$ P-a.s., and let $\mathbb{F}^X = (\mathcal{F}_t^X)_{t \geqslant 0}$ denote the (raw) filtration generated by X, i.e., $\mathcal{F}_t^X = \sigma(X_s;\, s \leqslant t)$. Show that the following two properties are equivalent:

(i) X has *independent increments*, i.e., for all $n \in \mathbb{N}$ and $0 \leqslant t_0 < t_1 < \cdots < t_n$, the increments $X_{t_i} - X_{t_{i-1}}$, $i = 1, \ldots, n$, are independent.

(ii) X has $\mathbb{F}^X$*-independent increments*, i.e., $X_t - X_s$ is independent of $\mathcal{F}_s^X$ whenever $t \geqslant s$.

Remark: This shows the equivalence of the properties (BM2) and (BM2′) of Brownian motion.

Hint: For proving "(i) ⇒ (ii)", you might use the monotone class theorem. When choosing the set $\mathcal{H}$, recall that a random variable Y is independent of a σ-algebra $\mathcal{G}$ if and only if $E[g(Y)Z] = E[g(Y)]E[Z]$ for all bounded measurable functions $g : \mathbb{R} \to \mathbb{R}$ and bounded $\mathcal{G}$-measurable random variables Z.

Solution 2.1 First, assume that (ii) holds. We use induction on n to show (i) holds. The base case $n = 1$ is trivial. Now fix $n \geqslant 2$, $0 \leqslant t_0 < t_1 < \cdots < t_n$ and $A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R})$. By (ii), we know that $X_{t_n} - X_{t_{n-1}}$ is independent of $\mathcal{F}_{t_{n-1}}^X$, and therefore

$$P\left[\bigcap_{i=1}^{n} \{X_{t_i} - X_{t_{i-1}} \in A_i\}\right] = P\left[\bigcap_{i=1}^{n-1} \{X_{t_i} - X_{t_{i-1}} \in A_i\}\right] P[X_{t_n} - X_{t_{n-1}} \in A_n].$$

By the induction hypothesis,

$$P\left[\bigcap_{i=1}^{n-1} \{X_{t_i} - X_{t_{i-1}} \in A_i\}\right] = \prod_{i=1}^{n-1} P[X_{t_i} - X_{t_{i-1}} \in A],$$

so that

$$P\left[\bigcap_{i=1}^{n} \{X_{t_i} - X_{t_{i-1}} \in A_i\}\right] = \prod_{i=1}^{n} P[X_{t_i} - X_{t_{i-1}} \in A],$$

as required.

Conversely, assume (i) holds and fix $0 \leqslant s \leqslant t$. Define the set $\mathcal{M}$ by

$$\mathcal{M} := \left\{ \prod_{i=1}^{n} h_i(X_{s_i}) : h_i : \mathbb{R} \to \mathbb{R} \text{ bdd measurable},\ 0 \leqslant s_1 < \cdots < s_n \leqslant s,\ n \in \mathbb{N} \right\}.$$

Then $\mathcal{M}$ is a family of bounded real-valued functions on Ω which is closed under multiplication. Moreover, we have $\sigma(\mathcal{M}) = \mathcal{F}_s^X$.

Now let $\mathcal{H}$ denote the set of all bounded and $\mathcal{F}_s^X$-measurable random variables Z satisfying

$$E[g(X_t - X_s)Z] = E[g(X_t - X_s)]E[Z], \quad \forall g : \mathbb{R} \to \mathbb{R} \text{ bounded measurable.}$$

Then $\mathcal{H}$ contains the constant function 1. By the dominated convergence theorem, we can see that $\mathcal{H}$ is also closed under (monotone) bounded convergence. To apply the monotone class theorem, it only remains to show $\mathcal{H} \supseteq \mathcal{M}$.

To this end, fix some $Z = \prod_{i=1}^{n} h_i(X_{s_i}) \in \mathcal{M}$ and define the measurable function $h : \mathbb{R}^n \to \mathbb{R}$ by $h(x_1, \ldots, x_n) = \prod_{i=1}^{n} h_i(x_i)$. Then we can write $Z = h(X_{s_1}, \ldots, X_{s_n})$. Remembering $X_0 = 0$ P-a.s., there exists some linear transformation $L : \mathbb{R}^n \to \mathbb{R}^n$ with

$$(X_{s_1}, \ldots, X_{s_n}) = L(X_{s_1} - X_0, X_{s_2} - X_{s_1}, \ldots, X_{s_n} - X_{s_{n-1}}) \quad P\text{-a.s.}$$

So we can write $Z = (h \circ L)(X_{s_1} - X_0, X_{s_2} - X_{s_1}, \ldots, X_{s_n} - X_{s_{n-1}})$. Since $X_t - X_s$ is independent of $(X_{s_1} - X_0, X_{s_2} - X_{s_1}, \ldots, X_{s_n} - X_{s_{n-1}})$ and $h \circ L$ is measurable, also $X_t - X_s$ is independent of Z. It then follows immediately that for any bounded measurable $g : \mathbb{R} \to \mathbb{R}$,

$$E[g(X_t - X_s)Z] = E[g(X_t - X_s)]E[Z].$$

We thus have $Z \in \mathcal{H}$, and as $Z \in \mathcal{M}$ was arbitrary, we have shown that $\mathcal{M} \subseteq \mathcal{H}$. So by the monotone class theorem, $\mathcal{H}$ contains every bounded $\mathcal{F}_s^X$-measurable random variable. From the hint, we conclude that $X_t - X_s$ is independent of $\mathcal{F}_s^X$. This completes the proof.

Exercise 2.2 *(Hölder continuity of Brownian paths)* For a fixed $\alpha > 0$, a function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is called *locally α-Hölder-continuous at a point $x \in D$* if there exist $\delta > 0$ and $C > 0$ such that $|f(x) - f(y)| \leqslant C|x - y|^\alpha$ for all $y \in D$ with $|x - y| \leqslant \delta$. If f is locally α-Hölder-continuous at every $x \in D$, we say that f is *locally α-Hölder-continuous.*

(a) Let $Z \sim \mathcal{N}(0, 1)$. Show that $P[|Z| \leqslant \varepsilon] \leqslant \varepsilon$ for any $\varepsilon \geqslant 0$.

(b) Let W be a Brownian motion. Prove that for any $\alpha > \frac{1}{2}$, P-almost all paths of W are nowhere locally α-Hölder-continuous on $[0, 1]$.

Hint: Take any $M \in \mathbb{N}$ satisfying $M(\alpha - \frac{1}{2}) > 1$ and show that the set $\{W_\cdot(\omega) \text{ is locally } \alpha\text{-Hölder at some } s \in [0, 1]\}$ is contained in the set $\bigcup_{C \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \bigcap_{n \geqslant m} \bigcup_{k=0,\ldots,n-M} \bigcap_{j=1}^{M} \{|W_{\frac{k+j}{n}}(\omega) - W_{\frac{k+j-1}{n}}(\omega)| \leqslant C \frac{1}{n^\alpha}\}$.

(c) The *Kolmogorov–Čentsov theorem* states that a stochastic process X on $[0,T]$ satisfying

$$E[|X_t - X_s|^\gamma] \leqslant C\,|t-s|^{1+\beta}, \quad s,t \in [0,T],$$

for some fixed $\gamma, \beta, C > 0$ has a version which is locally α-Hölder-continuous for each $\alpha < \beta/\gamma$. Use this result to deduce that Brownian motion is P-a.s. locally α-Hölder-continuous for every $\alpha < 1/2$.

Remark: One can also show that the Brownian paths are *not* locally 1/2-Hölder-continuous. The exact modulus of continuity was found by P. Lévy.

Solution 2.2

(a) We have

$$P[|Z| \leqslant \varepsilon] = P[-\varepsilon \leqslant Z \leqslant \varepsilon] = \int_{-\varepsilon}^{\varepsilon} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,\mathrm{d}x \leqslant \int_{-\varepsilon}^{\varepsilon} \frac{1}{\sqrt{2\pi}}\,\mathrm{d}x = \frac{2}{\sqrt{2\pi}}\varepsilon \leqslant \varepsilon,$$

as required.

(b) Fix $\alpha > \frac{1}{2}$ and let $M \in \mathbb{N}$ satisfy $M(\alpha - \frac{1}{2}) > 1$. Following the hint, suppose that $\omega \in \Omega$ is such that $W_\cdot(\omega)$ is locally α-Hölder-continuous at some $s \in [0,1]$. Then there exist $\delta > 0$ and $C \in \mathbb{N}$ such that $|W_t(\omega) - W_s(\omega)| \leqslant C|t-s|^\alpha$ for all $|t-s| < \delta$. Note that there is some $m \in \mathbb{N}$ large enough so that for all $n \geqslant m$, taking $k \in \{0, \dots, n-M\}$ minimal with $\frac{k}{n} > s - \delta$ gives $\frac{k+j}{n} \in (s-\delta, s+\delta)$ for all $j = 0, \dots, M$. It follows that the set

$$\{W_\cdot(\omega) \text{ is locally } \alpha\text{-Hölder at some } s \in [0,1]\}$$

is contained in the set

$$B := \bigcup_{C \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \bigcap_{n \geqslant m} \bigcup_{k=0,\dots,n-M} \bigcap_{j=1}^{M} \left\{ |W_{\frac{k+j}{n}}(\omega) - W_{\frac{k+j-1}{n}}(\omega)| \leqslant C \frac{1}{n^\alpha} \right\},$$

since for each $j = 1, \dots, M$, we have $|\frac{k+j}{n} - \frac{k+j-1}{n}| = \frac{1}{n}$. It thus suffices to show that B is a nullset.

Since the increments $W_{\frac{k+j}{n}} - W_{\frac{k+j-1}{n}}$, $j = 1, \dots, M$, are i.i.d. $\sim \mathcal{N}(0, \frac{1}{n})$, we can write, for $Z \sim \mathcal{N}(0,1)$,

$$P\left[\bigcap_{j=1}^{M} \left\{ |W_{\frac{k+j}{n}}(\omega) - W_{\frac{k+j-1}{n}}(\omega)| \leqslant C \frac{1}{n^\alpha} \right\} \right] = P\left[|Z| \leqslant \frac{C}{n^{\alpha - 1/2}} \right]^M.$$

Part (a) then gives

$$P\left[\bigcap_{j=1}^{M} \left\{ |W_{\frac{k+j}{n}}(\omega) - W_{\frac{k+j-1}{n}}(\omega)| \leqslant C \frac{1}{n^\alpha} \right\} \right] \leqslant C^M n^{-M(\alpha - \frac{1}{2})}. \tag{1}$$

Now if we set

$$D_m := \bigcap_{n\geqslant m} \bigcup_{k=0,\dots,n-M} \bigcap_{j=1}^{M} \left\{ |W_{\frac{k+j}{n}}(\omega) - W_{\frac{k+j-1}{n}}(\omega)| \leqslant C\frac{1}{n^\alpha} \right\},$$

then for each $n \geqslant m$, we have

$$D_m \subseteq \bigcup_{k=0,\dots,n-M} \bigcap_{j=1}^{M} \left\{ |W_{\frac{k+j}{n}}(\omega) - W_{\frac{k+j-1}{n}}(\omega)| \leqslant C\frac{1}{n^\alpha} \right\}.$$

So by using (4) and remembering that $M(\alpha - \frac{1}{2}) > 1$, we get

$$\begin{aligned} P[D_m] &\leqslant \limsup_{n\to\infty} P\left[\bigcup_{k=0,\dots,n-M} \bigcap_{j=1}^{M} \left\{ |W_{\frac{k+j}{n}}(\omega) - W_{\frac{k+j-1}{n}}(\omega)| \leqslant C\frac{1}{n^\alpha} \right\} \right] \\ &\leqslant \limsup_{n\to\infty} (n-M+1) C^M\, n^{-M(\alpha-\frac{1}{2})} \\ &= 0. \end{aligned}$$

As

$$B = \bigcup_{C\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} D_m$$

is a countable union of nullsets, we conclude that $P[B] = 0$, as claimed.

(c) Fix $0 \leqslant s < t$. We have $W_t - W_s \sim \mathcal{N}(0, t-s)$, and for each $n \in \mathbb{N}$,

$$E[|W_t - W_s|^{2n}] = |t-s|^n E[Z^{2n}] := C_{2n}|t-s|^n,$$

where $Z \sim \mathcal{N}(0,1)$ and $C_{2n} := E[Z^{2n}] < \infty$. Setting $\gamma_n := 2n$ and $\beta_n := n-1$, we can write

$$E[|W_t - W_s|^{\gamma_n}] = C_{2n}|t-s|^{1+\beta_n}.$$

Now fix $\alpha < \frac{1}{2}$. Since $\frac{\beta_n}{\gamma^n} \uparrow \frac{1}{2}$, there exists $N \in \mathbb{N}$ such that $\alpha < \frac{\beta_N}{\gamma_N}$. We can then apply the Kolmogorov–Čentsov theorem for this N to conclude that W has a locally α-Hölder-continuous version.

Now since both W and this locally α-Hölder-continuous version are continuous, we can apply Exercise 1.2(a) to conclude that they are in fact indistinguishable. Therefore, W itself is P-a.s. locally α-Hölder-continuous. Since a locally α-Hölder-continuous function is also locally β-Hölder-continuous whenever $\alpha > \beta$, we can conclude that W is P-a.s. locally α-Hölder-continuous for every $\alpha < \frac{1}{2}$.

Exercise 2.3 *(A new Brownian motion)* Let $(\Omega, \mathcal{F}, P)$ be a probability space, $W = (W_t)_{t\geqslant 0}$ a Brownian motion on $(\Omega, \mathcal{F}, P)$, Z a random variable independent of W and $s \in (0,\infty)$ a fixed time. We define the stochastic process $V = (V_t)_{t\geqslant 0}$ by

$$V_t := W_t \mathbf{1}_{\{t<s\}} + \Big(W_s + Z(W_t - W_s)\Big)\mathbf{1}_{\{t\geqslant s\}}.$$

Find all possible distributions of Z such that V is a Brownian motion.

Solution 2.3 First, we have $P[V_0 = 0] = P[W_0 = 0] = 1$, so that (BM1) always holds. Also, as $\lim_{t\downarrow s} V_t = \lim_{t\downarrow s}(W_s + Z(W_t - W_s)) = W_s = V_s$ P-a.s., we can see that (BM3) holds. Thus, V is a Brownian motion if and only if V satisfies (BM2′). We claim that this happens if and only if Z takes values in $\{-1, 1\}$ P-a.s.

To show this, fix $0 \leqslant t_0 < \cdots < t_k \leqslant s < t_{k+1} < \cdots < t_n$ and note that the random variables $V_{t_j} - V_{t_{j-1}}$, $j = 1, \dots, n$, are independent and with distribution $\mathcal{N}(0, t_j - t_{j-1})$ if and only if the characteristic function $\varphi : \mathbb{R}^n \to \mathbb{C}$ of the random vector $(V_{t_1} - V_{t_0}, \dots, V_{t_n} - V_{t_{n-1}})$ is equal to

$$\varphi(\lambda_1, \dots, \lambda_n) = \exp\bigg(-\frac{1}{2}\sum_{j=1}^{n} \lambda_j^2 (t_j - t_{j-1}) \bigg). \tag{2}$$

We can write

$$\begin{aligned}(V_{t_1} - V_0, \dots, V_{t_n} - V_{t_{n-1}}) = \big(&W_{t_1} - W_{t_0}, \dots, W_{t_k} - W_{t_{k-1}}, \\ &W_s + Z(W_{t_{k+1}} - W_s) - W_{t_k}, \\ &Z(W_{t_{k+2}} - W_{t_{k+1}}), \dots, Z(W_{t_n} - W_{t_{n-1}})\big).\end{aligned}$$

So we compute

$$\begin{aligned}\varphi(\lambda_1, \dots, \lambda_n) &:= E\bigg[\exp\bigg(i\sum_{j=1}^{n} \lambda_j (V_{t_j} - V_{t_{j-1}}) \bigg)\bigg] \\ &= E\bigg[\exp\bigg(i\sum_{j=1}^{k} \lambda_j (W_{t_j} - W_{t_{j-1}}) + i\lambda_{k+1}(W_s - W_{t_k}) \bigg) \\ &\qquad \times \exp\bigg(iZ\bigg(\lambda_{k+1}(W_{t_{k+1}} - W_s) + \sum_{j=k+2}^{n} \lambda_j (W_{t_j} - W_{t_{j-1}}) \bigg)\bigg)\bigg].\end{aligned}$$

From the independence of Brownian increments and the independence of W and Z, one can see that the two exponentials in the above product are independent, and so we have

$$\varphi(\lambda_1, \dots, \lambda_n) = E\bigg[\exp\bigg(i\sum_{j=1}^{k} \lambda_j (W_{t_j} - W_{t_{j-1}}) + i\lambda_{k+1}(W_s - W_{t_k}) \bigg)\bigg] \tag{3}$$

$$\times E\bigg[\exp\bigg(iZ\bigg(\lambda_{k+1}(W_{t_{k+1}} - W_s) + \sum_{j=k+2}^{n} \lambda_j (W_{t_j} - W_{t_{j-1}}) \bigg)\bigg)\bigg]. \tag{4}$$

Now using the fact that Brownian motion has independent Gaussian increments, we know (2) is equal to

$$\exp\bigg(-\frac{1}{2}\sum_{j=1}^{k} \lambda_j^2 (t_j - t_{j-1}) - \frac{1}{2}\lambda_{k+1}^2 (s - t_k) \bigg).$$

Also, we can write (3) as

$$E\bigg[E\bigg[\exp\bigg(iZ\Big(\lambda_{k+1}(W_{t_{k+1}}-W_s)+\sum_{j=k+2}^{n}\lambda_j(W_{t_j}-W_{t_{j-1}})\Big)\bigg)\ \bigg|\ Z\bigg]\bigg]$$
$$=E\bigg[\exp\bigg(-\frac{1}{2}Z^2\Big(\lambda_{k+1}^2(t_{k+1}-s)+\sum_{j=k+2}^{n}\lambda_j^2(t_j-t_{j-1})\Big)\bigg)\bigg].$$

It follows that if $Z^2=1$ P-a.s., then (1) holds. Conversely, suppose that (1) holds for all $n\in\mathbb{N}$, $\lambda_1,\dots,\lambda_n\in\mathbb{R}$ and $0\leqslant t_0<\dots<t_k\leqslant s<t_{k+1}<\dots<t_n$. To show that $Z^2=1$ P-a.s., we use that the Laplace transform $\rho\mapsto E[e^{-\rho Z^2}]$ (on $[0,\infty)$) of Z^2 is uniquely determined by its distribution. We start by choosing $n=1$, $t_0=0$ and $t_1>s$. Then the above calculations give that

$$\varphi(\lambda_1)=\exp\left(-\frac{1}{2}\lambda_1^2 s\right)E\left[\exp\left(-\frac{1}{2}Z^2\lambda_1^2(t_1-s)\right)\right],$$

while (1) gives

$$\varphi(\lambda_1)=\exp\left(-\frac{1}{2}\lambda_1^2 t_1\right).$$

It follows that

$$E\left[\exp\left(-\frac{1}{2}Z^2\lambda_1^2(t_1-s)\right)\right]=\exp\left(-\frac{1}{2}\lambda_1^2(t_1-s)\right).$$

Now set $\rho:=\frac{1}{2}\lambda_1^2(t_1-s)$. Since $t_1>s$, ρ will vary over all values of $[0,\infty)$ as λ_1 varies over $\mathbb{R}$. We thus have that for all $\rho\in[0,\infty)$,

$$E[\exp(-\rho Z^2)]=\exp(-\rho).$$

Hence, the Laplace transform of Z^2 is the same as the Laplace transform of the constant function 1, from which it follows that $Z^2=1$ P-a.s. This completes the proof.

Exercise 2.4 *(Blumenthal's 0-1 law)*

(a) Let W be a Brownian motion on a probability space $(\Omega,\mathcal{F},P)$ with natural filtration $(\mathcal{F}_t)_{t\geqslant 0}$, i.e. $\mathcal{F}_t=\sigma(W_s,0\leqslant s\leqslant t)$. Consider the σ-field

$$\mathcal{F}_{0+}:=\bigcap_{t>0}\mathcal{F}_t.$$

Establish *Blumenthal's 0-1 law*: for $A\in\mathcal{F}_{0+}$, either $P[A]=0$ or $P[A]=1$.

(b) Show that

$$P\left[\limsup_{t\downarrow 0}\frac{W_t}{\sqrt{t}}=\infty\right]=1.$$

Hint: Start by showing that for each $C > 0$,

$$\lim_{t \downarrow 0} P\left[\sup_{0 \leqslant s \leqslant t} (W_s - C\sqrt{s}) > 0\right] > 0$$

and then use part (a).

Solution 2.4

(a) We follow Le Gall, Theorem 2.13. Fix $A \in \mathcal{F}_{0+}$, $0 < t_1 < \cdots < t_k$ and let $g : \mathbb{R}^k \to \mathbb{R}$ be a bounded and continuous function. Since g is continuous and $W_\varepsilon \to 0$ P-a.s. as $\varepsilon \downarrow 0$, the dominated convergence theorem yields

$$E[\mathbf{1}_A g(W_{t_1}, \ldots, W_{t_k})] = \lim_{\varepsilon \downarrow 0} E[\mathbf{1}_A g(W_{t_1} - W_\varepsilon, \ldots, W_{t_k} - W_\varepsilon)].$$

Now for $0 < \varepsilon < t_1$, the random variables $W_{t_1} - W_\varepsilon, \ldots, W_{t_n} - W_\varepsilon$ are independent of $\mathcal{F}_\varepsilon$, and thus also of $A \in \mathcal{F}_{0+} \subseteq \mathcal{F}_\varepsilon$. So we can rewrite the above as

$$\begin{aligned} E[\mathbf{1}_A g(W_{t_1}, \ldots, W_{t_k})] &= P[A] \lim_{\varepsilon \downarrow 0} E[g(W_{t_1} - W_\varepsilon, \ldots, W_{t_k} - W_\varepsilon)] \\ &= P[A] E[g(W_{t_1}, \ldots, W_{t_k})], \end{aligned}$$

where in the last step we again use the dominated convergence theorem. It follows that $\mathcal{F}_{0+}$ is independent of $\sigma(W_{t_1}, \ldots, W_{t_k})$. As $0 < t_1 < \cdots < t_k$ were chosen arbitrarily and independence of σ-fields is equivalent to independence of π-systems that generate those σ-fields, we can conclude that $\mathcal{F}_{0+}$ is independent of $\sigma(W_t : t > 0)$. But as $W_0 = 0$, then of course $\sigma(W_t : t > 0) = \sigma(W_t : t \geqslant 0)$. Since $\mathcal{F}_{0+} \subseteq \sigma(W_t : t \geqslant 0)$, it follows that $\mathcal{F}_{0+}$ is independent of itself. This means that for each $A \in \mathcal{F}_{0+}$,

$$P[A] = P[A \cap A] = P[A]^2,$$

and thus $P[A] = 0$ or $P[A] = 1$, as required.

(b) Fix $C > 0$. For every $t > 0$, since $W_t \sim \mathcal{N}(0, t)$, we have

$$P[W_t > C\sqrt{t}] = 1 - \Phi(C),$$

where Φ denotes the cdf of the standard normal distribution. In particular, we have

$$\lim_{t \downarrow 0} P\left[\sup_{0 \leqslant s \leqslant t} (W_s - C\sqrt{s}) > 0\right] \geqslant \lim_{t \downarrow 0} P\left[W_t - C\sqrt{t} > 0\right] = 1 - \Phi(C) > 0.$$

Note also that

$$\left\{\limsup_{t \downarrow 0} \frac{W_t}{\sqrt{t}} \geqslant C\right\} = \bigcap_{n=1}^{\infty} \left\{\sup_{0 < t \leqslant \frac{1}{n}} \frac{W_t}{\sqrt{t}} \geqslant C\right\},$$

which shows in particular that $\{\limsup_{t\downarrow 0} \frac{W_t}{\sqrt{t}} \geqslant C\} \in \mathcal{F}_{0+}$ (as the sets $\{\sup_{0<t\leqslant \frac{1}{n}} \frac{W_t}{\sqrt{t}} \geqslant C\}$ are decreasing in $n \in \mathbb{N}$). We now write

$$\begin{aligned} P\left[\limsup_{t\downarrow 0} \frac{W_t}{\sqrt{t}} \geqslant C\right] &= P\left[\bigcap_{n=1}^{\infty} \left\{\sup_{0<t\leqslant \frac{1}{n}} \frac{W_t}{\sqrt{t}} \geqslant C\right\}\right] \\ &= \lim_{n\to\infty} P\left[\sup_{0<t\leqslant \frac{1}{n}} \frac{W_t}{\sqrt{t}} \geqslant C\right] \\ &\geqslant 1 - \Phi(C) \\ &> 0. \end{aligned}$$

So by Blumenthal's 0-1 law, it must be that

$$P\left[\limsup_{t\downarrow 0} \frac{W_t}{\sqrt{t}} \geqslant C\right] = 1.$$

Since

$$\left\{\limsup_{t\downarrow 0} \frac{W_t}{\sqrt{t}} = \infty\right\} = \bigcap_{m=1}^{\infty} \left\{\limsup_{t\downarrow 0} \frac{W_t}{\sqrt{t}} \geqslant m\right\},$$

the claim follows.