Brownian Motion and Stochastic Calculus

Exercise Sheet 3

Submit by 12:00 on Wednesday, March 12 via the course homepage.

Exercise 3.1 *(Ornstein–Uhlenbeck process)* Let $W = (W_t)_{t \geqslant 0}$ be a Brownian motion and consider the $\mathbb{R}$-indexed stochastic process $X = (X_t)_{t \in \mathbb{R}}$ defined by

$$X_t := e^{-t} W_{e^{2t}}.$$

The process X is called an *Ornstein–Uhlenbeck process.*

(a) For fixed $t \in \mathbb{R}$, what is the distribution of X_t?

(b) Show that the process $(X_t)_{t \in \mathbb{R}}$ is *time-reversible*, meaning that

$$(X_t)_{t \geqslant 0} \overset{(d)}{=} (X_{-t})_{t \geqslant 0}.$$

Note that the equality above means that the distribution of the left-hand side is the same as the distribution of the right-hand side, *as stochastic processes.* This says more than simply having $X_t \overset{(d)}{=} X_{-t}$ for each $t \geqslant 0$.

Exercise 3.2 *(Non-adapted process)* Let $W = (W_t)_{0 \leqslant t \leqslant 1}$ be a Brownian motion on $[0, 1]$ with respect to its natural filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leqslant t \leqslant 1}$. Consider the stochastic process $X = (X_t)_{0 \leqslant t \leqslant 1}$ given by

$$X_t := x(1 - t) + yt + (W_t - tW_1),$$

where $x, y \in \mathbb{R}$ are fixed constants.

(a) Show that X is a continuous Gaussian process with $X_0 = x$ and $X_1 = y$. The process X is also called the *Brownian bridge* from x to y over $[0, 1]$.

(b) Calculate the mean and covariance function of $(X_t)_{0 \leqslant t \leqslant 1}$.

(c) Show that X is not $\mathbb{F}$-adapted.

(d) Let $\mathbb{F}^X = (\mathcal{F}_t^X)_{0 \leqslant t \leqslant 1}$ denote the natural filtration of X. Is W also a Brownian motion on $[0, 1]$ with respect to $\mathbb{F}^X$?

Exercise 3.3 *(Martingales)* Let W be a Brownian motion with respect to its natural filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geqslant 0}$.

(a) Show that the process $M = (M_t)_{t\geqslant 0}$ given by

$$M_t = tW_t - \int_0^t W_u \,\mathrm{d}u,$$

is a martingale, under the assumption that the filtration $\mathbb{F}$ is complete.

(b) Show that the process $N = (N_t)_{t\geqslant 0}$ given by

$$N_t = W_t^3 - 3tW_t,$$

is a martingale.

Exercise 3.4 *(σ-field of the past before τ)* Given a measurable space $(\Omega, \mathcal{F})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$, we set $\mathcal{F}_\infty := \sigma(\bigcup_{t\geqslant 0} \mathcal{F}_t)$ and define for any $\mathbb{F}$-stopping time τ the σ-field

$$\mathcal{F}_\tau := \Big\{A \in \mathcal{F}_\infty : A \cap \{\tau \leqslant t\} \in \mathcal{F}_t \text{ for all } t \geqslant 0\Big\}.$$

Let S, T be $\mathbb{F}$-stopping times. Show the following:

(a) If $S \leqslant T$, then $\mathcal{F}_S \subseteq \mathcal{F}_T$.

(b) $\mathcal{F}_{S\wedge T} = \mathcal{F}_S \cap \mathcal{F}_T$.

(c) For any $A \in \mathcal{F}_S$, both $A \cap \{S < T\}$ and $A \cap \{S \leqslant T\}$ are in $\mathcal{F}_{S\wedge T}$.

Note that this shows in particular that $\{S < T\}$, $\{S \leqslant T\} \in \mathcal{F}_{S\wedge T}$.

(d) For any stopping time τ,

$$\mathcal{F}_\tau = \sigma(X_\tau : X \text{ is an optional process}).$$