Brownian Motion and Stochastic Calculus

Exercise Sheet 3

Submit by 12:00 on Wednesday, March 12 via the course homepage.

Exercise 3.1 *(Ornstein–Uhlenbeck process)* Let $W = (W_t)_{t\geqslant 0}$ be a Brownian motion and consider the $\mathbb{R}$-indexed stochastic process $X = (X_t)_{t\in\mathbb{R}}$ defined by

$$X_t := e^{-t} W_{e^{2t}}.$$

The process X is called an *Ornstein–Uhlenbeck process*.

(a) For fixed $t \in \mathbb{R}$, what is the distribution of X_t?

(b) Show that the process $(X_t)_{t\in\mathbb{R}}$ is *time-reversible*, meaning that

$$(X_t)_{t\geqslant 0} \overset{(\mathrm{d})}{=} (X_{-t})_{t\geqslant 0}.$$

Note that the equality above means that the distribution of the left-hand side is the same as the distribution of the right-hand side, *as stochastic processes*. This says more than simply having $X_t \overset{(\mathrm{d})}{=} X_{-t}$ for each $t \geqslant 0$.

Solution 3.1

(a) We know that $W_{e^{2t}} \sim \mathcal{N}(0, e^{2t})$, and therefore $X_t \sim \mathcal{N}(0, 1)$.

(b) It is enough to show that the finite-dimensional distributions of $(X_t)_{t\geqslant 0}$ and $(X_{-t})_{t\geqslant 0}$ agree. To this end, fix $n \in \mathbb{N}$ and $t_1, t_2, \dots, t_n > 0$ (in the case that one of the t_i is zero there is nothing to do). We need to show that

$$(X_{t_1}, X_{t_2}, \dots, X_{t_n}) \overset{(\mathrm{d})}{=} (X_{-t_1}, X_{-t_2}, \dots, X_{-t_n}).$$

We first write

$$(X_{-t_1}, \dots, X_{-t_n}) = (e^{t_1} W_{e^{-2t_1}}, \dots, e^{t_n} W_{e^{-2t_n}}).$$

By the time inversion property of Brownian motion in Proposition 2.1.1(4),

$$(W_{e^{-2t_1}}, \dots, W_{e^{-2t_n}}) \overset{(\mathrm{d})}{=} (e^{-2t_1} W_{e^{2t_1}}, \dots, e^{-2t_n} W_{e^{2t_n}}),$$

and thus, writing $\cdot$ for the coordinatewise product,

$$(e^{t_1}, \dots, e^{t_n}) \cdot (W_{e^{-2t_1}}, \dots, W_{e^{-2t_n}}) \overset{(\mathrm{d})}{=} (e^{t_1}, \dots, e^{t_n}) \cdot (e^{-2t_1} W_{e^{2t_1}}, \dots, e^{-2t_n} W_{e^{2t_n}}).$$

It follows that

$$(X_{-t_1}, \dots, X_{-t_n}) \overset{\text{(d)}}{=} (e^{-t_1} W_{e^{2t_1}}, \dots, e^{-t_n} W_{e^{2t_n}}) = (X_{t_1}, \dots, X_{t_n}),$$

as required.

Exercise 3.2 *(Non-adapted process)* Let $W = (W_t)_{0 \leqslant t \leqslant 1}$ be a Brownian motion on $[0,1]$ with respect to its natural filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leqslant t \leqslant 1}$. Consider the stochastic process $X = (X_t)_{0 \leqslant t \leqslant 1}$ given by

$$X_t := x(1-t) + yt + (W_t - tW_1),$$

where $x, y \in \mathbb{R}$ are fixed constants.

(a) Show that X is a continuous Gaussian process with $X_0 = x$ and $X_1 = y$. The process X is also called the *Brownian bridge* from x to y over $[0,1]$.

(b) Calculate the mean and covariance function of $(X_t)_{0 \leqslant t \leqslant 1}$.

(c) Show that X is not $\mathbb{F}$-adapted.

(d) Let $\mathbb{F}^X = (\mathcal{F}_t^X)_{0 \leqslant t \leqslant 1}$ denote the natural filtration of X. Is W also a Brownian motion on $[0,1]$ with respect to $\mathbb{F}^X$?

Solution 3.2

(a) It is immediate that $X_0 = x$ (since $W_0 = 0$) and $X_1 = y$. Also, X is continuous since W is continuous on $[0,1]$. To see that X is a Gaussian process, fix $n \in \mathbb{N}$ and choose $0 \leqslant t_0 < \cdots < t_n \leqslant 1$ and $\lambda_0, \dots \lambda_n \in \mathbb{R}$. We compute

$$\begin{aligned} \lambda_0 X_{t_0} + \cdots + \lambda_n X_{t_n} = (\lambda_0 + \cdots + \lambda_n)x + (y-x)(\lambda_0 t_0 + \cdots + \lambda_n t_n) \\ + \lambda_0 W_{t_0} + \cdots + \lambda_n W_{t_n} - (\lambda_0 t_0 + \cdots + \lambda_n t_n) W_1. \end{aligned}$$

In particular, $\lambda_0 X_{t_0} + \cdots + \lambda_n X_{t_n}$ is the sum of a constant and a linear combination of the random variables $W_{t_0}, \dots, W_{t_n}, W_1$. It is thus also the sum of a constant and a linear combination of $W_1 - W_{t_n}, W_{t_n} - W_{t_{n-1}}, \dots, W_{t_1} - W_{t_0}$. Since $W_1 - W_{t_n}, W_{t_n} - W_{t_{n-1}}, \dots, W_{t_1} - W_{t_0}$ are independent normal random variables, their linear combination is normal, and thus also $\lambda_0 X_{t_0} + \cdots + \lambda_n X_{t_n}$ is normal. So X is a Gaussian process, as required.

(b) Fix $0 \leqslant s \leqslant t \leqslant 1$. We have

$$E[X_t] = x(1-t) + yt + E[W_t] - tE[W_1] = x(1-t) + yt,$$

and

$$\begin{aligned} \operatorname{Cov}(X_s, X_t) &= \operatorname{Cov}(W_s - sW_1, W_t - tW_1) \\ &= s - ts - st + st \\ &= s - st \\ &= s(1-t). \end{aligned}$$

(c) Suppose for contradiction that X is $\mathbb{F}$-adapted. Then in particular, $X_{\frac{1}{2}}$ is $\mathcal{F}_{\frac{1}{2}}$-measurable. We have

$$\begin{aligned} X_{\frac{1}{2}} &= E[X_{\frac{1}{2}} \mid \mathcal{F}_{\frac{1}{2}}] \\ &= \frac{1}{2}(x+y) + E\left[W_{\frac{1}{2}} - \frac{1}{2}W_1 \,\middle|\, \mathcal{F}_{\frac{1}{2}}\right] \\ &= \frac{1}{2}(x+y) - \frac{1}{2}E\left[W_1 - W_{\frac{1}{2}} \,\middle|\, \mathcal{F}_{\frac{1}{2}}\right] + \frac{1}{2}E[W_{\frac{1}{2}} \mid \mathcal{F}_{\frac{1}{2}}] \\ &= \frac{1}{2}W_{\frac{1}{2}} + \frac{1}{2}(x+y). \end{aligned}$$

But $X_{\frac{1}{2}} = \frac{1}{2}(x+y) + W_{\frac{1}{2}} - \frac{1}{2}W_1$ so that we obtain

$$\frac{1}{2}(W_1 - W_{\frac{1}{2}}) = 0 \quad P\text{-a.s.},$$

which contradicts $W_1 - W_{\frac{1}{2}} \sim \mathcal{N}(0, \frac{1}{2})$. This completes the proof.

(d) Suppose for contradiction that W is an $\mathbb{F}^X$-Brownian motion. Then in particular $W_{\frac{1}{2}}$ is $\mathcal{F}^X_{\frac{1}{2}}$-measurable. Since $X_{\frac{1}{2}}$ is $\mathcal{F}^X_{\frac{1}{2}}$-measurable and

$$X_{\frac{1}{2}} = \frac{1}{2}(x+y) + W_{\frac{1}{2}} - \frac{1}{2}W_1,$$

we obtain that $W_{\frac{1}{2}} - \frac{1}{2}W_1$ is $\mathcal{F}^X_{\frac{1}{2}}$-measurable. Thus $W_1 - W_{\frac{1}{2}} = W_{\frac{1}{2}} - 2(W_{\frac{1}{2}} - \frac{1}{2}W_1)$ is $\mathcal{F}^X_{\frac{1}{2}}$-measurable. But from the assumption that W is an $\mathbb{F}^X$-Brownian motion, we also have that $W_1 - W_{\frac{1}{2}}$ is independent of $\mathcal{F}^X_{\frac{1}{2}}$, and thus $W_1 - W_{\frac{1}{2}}$ is independent of itself. This means that $W_1 - W_{\frac{1}{2}}$ is almost surely constant, which contradicts $W_1 - W_{\frac{1}{2}} \sim \mathcal{N}(0, \frac{1}{2})$. Thus W is not a Brownian motion with respect to $\mathbb{F}^X$.

Exercise 3.3 *(Martingales)* Let W be a Brownian motion with respect to its natural filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geqslant 0}$.

(a) Show that the process $M = (M_t)_{t \geqslant 0}$ given by

$$M_t = tW_t - \int_0^t W_u \, \mathrm{d}u,$$

is a martingale, under the assumption that the filtration $\mathbb{F}$ is complete.

(b) Show that the process $N = (N_t)_{t \geqslant 0}$ given by

$$N_t = W_t^3 - 3tW_t,$$

is a martingale.

Solution 3.3

(a) First, we can see that M is adapted to $\mathbb{F}$. Indeed, for fixed $t > 0$ the integral $\int_0^t W_u \,\mathrm{d}u$ is the almost-sure limit of left-hand Riemann sums, which are each $\mathcal{F}_t$-measurable. As $\mathcal{F}_t$ is complete, this implies that $\int_0^t W_u \,\mathrm{d}u$ is also $\mathcal{F}_t$-measurable (the full details are left as an exercise). Next, we have

$$\begin{aligned} E[|M_t|] &\leqslant E[t|W_t|] + E\left[\int_0^t |W_u| \,\mathrm{d}u\right] = tE[|W_t|] + \int_0^t E[|W_u|] \,\mathrm{d}u \\ &= \frac{t\sqrt{2t}}{\sqrt{\pi}} + \int_0^t \frac{\sqrt{2u}}{\sqrt{\pi}} \,\mathrm{d}u < \infty \end{aligned}$$

so that M is integrable. It remains to check that M satisfies the martingale property. For $0 \leqslant s \leqslant t$, we have

$$\begin{aligned} E[M_t \mid \mathcal{F}_s] &= tW_s - \int_0^s W_u \,\mathrm{d}u - E\left[\int_s^t W_u \,\mathrm{d}u \,\middle|\, \mathcal{F}_s\right] \\ &= tW_s - \int_0^s W_u \,\mathrm{d}u - \int_s^t E[W_u \mid \mathcal{F}_s] \,\mathrm{d}u \\ &= tW_s - \int_0^s W_u \,\mathrm{d}u - (t-s)W_s \\ &= M_s, \end{aligned}$$

where we use that $E[W_u \mid \mathcal{F}_s] = W_s$ for $u \geqslant s$. This completes the proof.

(b) It is clear that N is adapted. Moreover, N is integrable because the normal distribution has finite moments. Finally, to show that N satisfies the martingale property, we fix $0 \leqslant s \leqslant t$ and write

$$\begin{aligned} E[N_t \mid \mathcal{F}_s] &= E[W_t^3 \mid \mathcal{F}_s] - 3tW_s \\ &= E[(W_s + W_t - W_s)^3 \mid \mathcal{F}_s] - 3tW_s \\ &= E[W_s^3 + 3W_s^2(W_t - W_s) + 3W_s(W_t - W_s)^2 + (W_t - W_s)^3 \mid \mathcal{F}_s] \\ &\quad - 3tW_s \\ &= W_s^3 + 3W_s^2 E[W_t - W_s] + 3W_s E[(W_t - W_s)^2] + E[(W_t - W_s)^3] \\ &\quad - 3tW_s \\ &= W_s^3 + 3W_s(t-s) - 3tW_s \\ &= W_s^3 - 3sW_s \\ &= N_s, \end{aligned}$$

as required.

Exercise 3.4 *(σ-field of the past before τ)* Given a measurable space $(\Omega, \mathcal{F})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$, we set $\mathcal{F}_\infty := \sigma(\bigcup_{t\geqslant 0} \mathcal{F}_t)$ and define for any $\mathbb{F}$-stopping time τ the σ-field

$$\mathcal{F}_\tau := \Big\{A \in \mathcal{F}_\infty : A \cap \{\tau \leqslant t\} \in \mathcal{F}_t \ \text{ for all } t \geqslant 0\Big\}.$$

Let S, T be $\mathbb{F}$-stopping times. Show the following:

(a) If $S \leqslant T$, then $\mathcal{F}_S \subseteq \mathcal{F}_T$.

(b) $\mathcal{F}_{S\wedge T} = \mathcal{F}_S \cap \mathcal{F}_T$.

(c) For any $A \in \mathcal{F}_S$, both $A \cap \{S < T\}$ and $A \cap \{S \leqslant T\}$ are in $\mathcal{F}_{S\wedge T}$.

Note that this shows in particular that $\{S < T\}$, $\{S \leqslant T\} \in \mathcal{F}_{S\wedge T}$.

(d) For any stopping time τ,

$$\mathcal{F}_\tau = \sigma(X_\tau : X \text{ is an optional process}).$$

Solution 3.4

(a) Let $A \in \mathcal{F}_S$. For $t \geqslant 0$, we have $\{T \leqslant t\} \subseteq \{S \leqslant t\}$ since $S \leqslant T$. Thus

$$A \cap \{T \leqslant t\} = (A \cap \{S \leqslant t\}) \cap \{T \leqslant t\}.$$

But $A \cap \{S \leqslant t\} \in \mathcal{F}_t$ as $A \in \mathcal{F}_S$, while $\{T \leqslant t\} \in \mathcal{F}_t$ as T is a stopping time. Thus $A \cap \{T \leqslant t\} \in \mathcal{F}_t$ and so $A \in \mathcal{F}_T$. As $A \in \mathcal{F}_S$ was chosen arbitrarily, we have shown that $\mathcal{F}_S \subseteq \mathcal{F}_T$.

(b) Note that $S \wedge T$ is a stopping time with $S \wedge T \leqslant S$ and $S \wedge T \leqslant T$; so by part (a), we can conclude that $\mathcal{F}_{S\wedge T} \subseteq \mathcal{F}_S \cap \mathcal{F}_T$. Conversely, take $A \in \mathcal{F}_S \cap \mathcal{F}_T$ and note that

$$A \cap \{S \wedge T \leqslant t\} = (A \cap \{S \leqslant t\}) \cup (A \cap \{T \leqslant t\}).$$

Since $A \cap \{S \leqslant t\}, A \cap \{T \leqslant t\} \in \mathcal{F}_t$, we have $A \cap \{S \wedge T \leqslant t\} \in \mathcal{F}_t$, so that $\mathcal{F}_S \cap \mathcal{F}_T \subseteq \mathcal{F}_{S\wedge T}$. This completes the proof.

(c) We first show that $\{S < T\}, \{S \leqslant T\} \in \mathcal{F}_{S\wedge T}$. By part (b), it suffices to show that each of these sets belong to both $\mathcal{F}_S$ and $\mathcal{F}_T$. To this end, fix $t \geqslant 0$ and write

$$\{S < T\} \cap \{T \leqslant t\} = \left(\bigcup_{q \in \mathbb{Q} \cap [0,t]} \{S \leqslant q < T\} \right) \cap \{T \leqslant t\}.$$

Since $\{S \leqslant q < T\} = \{S \leqslant q\} \cap \{q < T\} \in \mathcal{F}_q \subseteq \mathcal{F}_t$ and $\{T \leqslant t\} \in \mathcal{F}_t$, it follows that $\{S < T\} \cap \{T \leqslant t\} \in \mathcal{F}_t$ (since the above union is countable). We have thus shown that $\{S < T\} \in \mathcal{F}_T$. Next, we write

$$\begin{aligned}\{S < T\} \cap \{S \leqslant t\} = (&\{S < T\} \cap \{S \leqslant t\} \cap \{t < T\}) \\ &\cup (\{S < T\} \cap \{S \leqslant t\} \cap \{T \leqslant t\}) \\ = (&\{S \leqslant t\} \cap \{t < T\}) \cup (\{S < T\} \cap \{T \leqslant t\}).\end{aligned}$$

By the above we know that $\{S < T\} \cap \{T \leqslant t\} \in \mathcal{F}_t$. Moreover we clearly have $\{S \leqslant t\} \cap \{t < T\} \in \mathcal{F}_t$ and thus $\{S < T\} \cap \{S \leqslant t\} \in \mathcal{F}_t$, so that $\{S < T\} \in \mathcal{F}_S$. Hence $\{S < T\} \in \mathcal{F}_{S \wedge T}$, as claimed.

Next, note that $\{S \leqslant T\} = \{T < S\}^c$ and by symmetry $\{T < S\} \in \mathcal{F}_{S \wedge T}$, so that also $\{S \leqslant T\} \in \mathcal{F}_{S \wedge T}$.

Now fix $A \in \mathcal{F}_S$. For any $t \geqslant 0$, we have

$$\begin{aligned} A \cap \{S < T\} \cap \{S \wedge T \leqslant t\} &= A \cap \{S < T\} \cap \{S \leqslant t\} \\ &= (A \cap \{S \leqslant t\}) \cap (\{S < T\} \cap \{S \leqslant t\}). \end{aligned}$$

Since $A \in \mathcal{F}_S$, we have $A \cap \{S \leqslant t\} \in \mathcal{F}_t$. Moreover, since $\{S < T\} \in \mathcal{F}_S$ by above, we also have $\{S \leqslant T\} \cap \{S \leqslant t\} \in \mathcal{F}_t$, and thus we can conclude that $A \cap \{S < T\} \cap \{S \wedge T \leqslant t\} \in \mathcal{F}_t$, so that $A \cap \{S < T\} \in \mathcal{F}_{S \wedge T}$.

Finally, we can similarly write

$$A \cap \{S \leqslant T\} \cap \{S \wedge T \leqslant t\} = (A \cap \{S \leqslant t\}) \cap (\{S \leqslant T\} \cap \{S \leqslant t\}) \in \mathcal{F}_t,$$

implying that $A \cap \{S \leqslant T\} \in \mathcal{F}_{S \wedge T}$. This completes the proof.

(d) First we show the inclusion "$\subseteq$". So take $A \in \mathcal{F}_\tau$ and define the process $X = (X_t)_{t \geqslant 0}$ by $X_t := \mathbf{1}_{A \cap \{\tau \leqslant t\}}$. Then X is adapted since $A \cap \{\tau \leqslant t\} \in \mathcal{F}_t$. Moreover, X is RCLL (with a single jump at time τ). In particular, X is optional. Since $X_\tau = \mathbf{1}_A$, this shows the inclusion "$\subseteq$".

Conversely, let X be an optional process. We first assume that X is right-continuous and adapted. To show that X_τ is $\mathcal{F}_\tau$-measurable, it suffices to show that $\{X_\tau \in U\} \in \mathcal{F}_\tau$ for all open $U \subseteq \mathbb{R}$. So take an open subset $U \subseteq \mathbb{R}$ and fix $t \geqslant 0$. By right-continuity of X, we have on $\{\tau < t\}$ that $X_\tau \in U$ if and only if there exists some $\varepsilon > 0$ (depending on ω) such that $\inf_{s \in (\tau, \tau+\varepsilon) \cap [0,t]} \operatorname{dist}(X_s, U^c) \geqslant \varepsilon$.

So we have

$$\begin{aligned} &\{X_\tau \in U\} \cap \{\tau \leqslant t\} = \\ &\quad \big(\{X_t \in B\} \cap \{\tau = t\}\big) \\ &\quad \cup \left(\bigcup_{\varepsilon \in \mathbb{Q}_{++}} \bigcap_{s \in \mathbb{Q} \cap [0,t]} \Big(\{\tau < s\} \cap \{s < \tau + \varepsilon\} \cap \{\operatorname{dist}(X_s, B^c) < \varepsilon\} \Big) \right) \in \mathcal{F}_t. \end{aligned}$$

Therefore, X_τ is $\mathcal{F}_\tau$-measurable, as required.

It remains to show that X_τ is $\mathcal{F}_\tau$-measurable for a general optional process X. To this end, we argue using the monotone class theorem. Let $\mathcal{M}$ be the set of adapted, bounded and right-continuous processes and $\mathcal{H}$ the set of bounded processes such that X_τ is $\mathcal{F}_\tau$-measurable. It is clear that $\mathcal{M}$ is closed under multiplication and $\sigma(\mathcal{M}) = \mathcal{O}$. Moreover, $\mathcal{H}$ is a real vector space such that $1 \in \mathcal{M} \subseteq \mathcal{H}$ (by the previous argument). Moreover, $\mathcal{H}$ is closed under bounded monotone convergence, since if $X^n \uparrow X$ with X bounded and each $X^n \in \mathcal{H}$, then each X^n_τ is $\mathcal{F}_\tau$-measurable and thus also $X_\tau = \lim_{n\to\infty} X^n_\tau$ is $\mathcal{F}_\tau$-measurable. So by the monotone class theorem, $\mathcal{H}$ contains all bounded optional processes. In other words, X_τ is $\mathcal{F}_\tau$-measurable for all bounded optional X.

Finally, if X is unbounded, we simply define $X^n := X\mathbf{1}_{\{|X|\leqslant n\}}$, $n \in \mathbb{N}$, which are bounded and optional. Therefore X^n_τ is $\mathcal{F}_\tau$-measurable for each $n \in \mathbb{N}$. As $X_\tau = \lim_{n\to\infty} X^n_\tau$, also X_τ is $\mathcal{F}_\tau$-measurable. This completes the proof.