Coordinator
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Brownian Motion and Stochastic Calculus

Exercise Sheet 4

Submit by 12:00 on Wednesday, March 19 via the course homepage.

Exercise 4.1 *(Commutativity of conditioning on stopping time σ-fields)* Consider two stopping times σ, τ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, P)$. The aim of this exercise is to show that for all integrable random variables Z,

$$E\Big[E[Z \,|\, \mathcal{F}_\sigma] \,\Big|\, \mathcal{F}_\tau\Big] = E\Big[E[Z \,|\, \mathcal{F}_\tau] \,\Big|\, \mathcal{F}_\sigma\Big] = E[Z \,|\, \mathcal{F}_{\sigma \wedge \tau}], \tag{$*$}$$

i.e., the operators $E[\,\cdot\,|\mathcal{F}_\sigma]$ and $E[\,\cdot\,|\mathcal{F}_\tau]$ on $L^1(\Omega)$ commute and their composition is $E[\,\cdot\,|\mathcal{F}_{\sigma\wedge\tau}]$.

Remark: For arbitrary sub-σ-algebras $\mathcal{G}, \mathcal{G}' \subseteq \mathcal{F}$, the conditional expectations $E[E[\,\cdot\,|\mathcal{G}]|\mathcal{G}']$, $E[E[\,\cdot\,|\mathcal{G}']|\mathcal{G}]$ and $E[\,\cdot\,|\mathcal{G} \cap \mathcal{G}']$ do not coincide in general.

(a) Show that if Y is an $\mathcal{F}_\sigma$-measurable random variable, then $Y\mathbf{1}_{\{\sigma \leqslant \tau\}}$ and $Y\mathbf{1}_{\{\sigma < \tau\}}$ are $\mathcal{F}_{\sigma\wedge\tau}$-measurable.

(b) Show that if Y is an $\mathcal{F}_\sigma$-measurable and integrable random variable, then $E[Y \,|\, \mathcal{F}_\tau]$ is $\mathcal{F}_{\sigma\wedge\tau}$-measurable.

(c) Deduce $(*)$.

Exercise 4.2 *(Stopped martingales)* Let $M = (M_t)_{t \geqslant 0}$ be a martingale with right-continuous sample paths and let τ be a stopping time with respect to the same filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geqslant 0}$. Define the *stopped process* $M^\tau = (M^\tau_t)_{t \geqslant 0}$ by

$$M^\tau_t := M_{t \wedge \tau}.$$

(a) Suppose additionally that M is uniformly integrable. Show that for each $t \geqslant 0$,

$$M^\tau_t = E[M_\tau \,|\, \mathcal{F}_t].$$

Deduce that M^τ is a uniformly integrable martingale.

(b) Without assuming that M is uniformly integrable, show that the stopped process M^τ is still a martingale.

Exercise 4.3 *(Ruin problem for Brownian motion)* Let $W = (W_t)_{t \geqslant 0}$ be a Brownian motion. For each $x \in \mathbb{R}$, define the stopping time τ_x by

$$\tau_x := \inf\{t \geqslant 0 : W_t = x\}.$$

Fix $a < 0 < b$ and set $\tau := \tau_a \wedge \tau_b$.

(a) Show that for each $\lambda > 0$,

$$E[e^{-\lambda\tau}] = \frac{\cosh(\frac{b+a}{2}\sqrt{2\lambda})}{\cosh(\frac{b-a}{2}\sqrt{2\lambda})}.$$

Hint: For a suitable choice of $\alpha \in \mathbb{R}$, consider the process $M = (M_t)_{t\geqslant 0}$ given by

$$M_t := e^{\sqrt{2\lambda}(W_t-\alpha)-\lambda t} + e^{-\sqrt{2\lambda}(W_t-\alpha)-\lambda t}.$$

You may want to think about why M is a martingale.

(b) Show similarly that for every $\lambda > 0$,

$$E[e^{-\lambda\tau}\mathbf{1}_{\{\tau=\tau_a\}}] = \frac{\sinh(b\sqrt{2\lambda})}{\sinh((b-a)\sqrt{2\lambda})}.$$

(c) Find the value of $P[\tau_a < \tau_b]$.

Hint: You may use the identity

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y).$$