Brownian Motion and Stochastic Calculus

Exercise Sheet 4

Submit by 12:00 on Wednesday, March 19 via the course homepage.

Exercise 4.1 *(Commutativity of conditioning on stopping time σ-fields)* Consider two stopping times σ, τ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, P)$. The aim of this exercise is to show that for all integrable random variables Z,

$$E\Big[E[Z \mid \mathcal{F}_\sigma] \Big| \mathcal{F}_\tau\Big] = E\Big[E[Z \mid \mathcal{F}_\tau] \Big| \mathcal{F}_\sigma\Big] = E[Z \mid \mathcal{F}_{\sigma\wedge\tau}], \tag{$*$}$$

i.e., the operators $E[\,\cdot\,|\mathcal{F}_\sigma]$ and $E[\,\cdot\,|\mathcal{F}_\tau]$ on $L^1(\Omega)$ commute and their composition is $E[\,\cdot\,|\mathcal{F}_{\sigma\wedge\tau}]$.

Remark: For arbitrary sub-σ-algebras $\mathcal{G}, \mathcal{G}' \subseteq \mathcal{F}$, the conditional expectations $E[E[\,\cdot\,|\mathcal{G}]|\mathcal{G}']$, $E[E[\,\cdot\,|\mathcal{G}']|\mathcal{G}]$ and $E[\,\cdot\,|\mathcal{G}\cap\mathcal{G}']$ do not coincide in general.

(a) Show that if Y is an $\mathcal{F}_\sigma$-measurable random variable, then $Y\mathbf{1}_{\{\sigma\leqslant\tau\}}$ and $Y\mathbf{1}_{\{\sigma<\tau\}}$ are $\mathcal{F}_{\sigma\wedge\tau}$-measurable.

(b) Show that if Y is an $\mathcal{F}_\sigma$-measurable and integrable random variable, then $E[Y \mid \mathcal{F}_\tau]$ is $\mathcal{F}_{\sigma\wedge\tau}$-measurable.

(c) Deduce ($*$).

Solution 4.1

(a) Since $\{\sigma\leqslant\tau\}, \{\sigma<\tau\} \in \mathcal{F}_\sigma$ by Exercise 3.4, we have that $Y\mathbf{1}_{\{\sigma\leqslant\tau\}}, Y\mathbf{1}_{\{\sigma<\tau\}}$ are both $\mathcal{F}_\sigma$-measurable. To show $\mathcal{F}_{\sigma\wedge\tau}$-measurability, we only argue for $Y\mathbf{1}_{\{\sigma\leqslant\tau\}}$, as the argument for $Y\mathbf{1}_{\{\sigma<\tau\}}$ is analogous. First, consider the case that Y is a simple function, say $Y = \sum_{i=1}^n \lambda_i \mathbf{1}_{A_i}$ for some $A_1, \dots, A_n \in \mathcal{F}_\sigma$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. Then we have $Y\mathbf{1}_{\{\sigma\leqslant\tau\}} = \sum_{i=1}^n \lambda_i \mathbf{1}_{A_i\cap\{\sigma\leqslant\tau\}}$. By Exercise 3.4, $A_i \cap \{\sigma\leqslant\tau\} \in \mathcal{F}_{\sigma\wedge\tau}$ for each $1\leqslant i\leqslant n$, and thus Y is $\mathcal{F}_{\sigma\wedge\tau}$-measurable.

For general Y, we can construct simple random variables Y^n of the above form such that $Y^n(\omega) \to Y(\omega)$ for all $\omega\in\Omega$ (under the assumption of completeness of $\mathcal{F}_0$), and thus $Y^n\mathbf{1}_{\{\sigma\leqslant\tau\}} \to Y\mathbf{1}_{\{\sigma\leqslant\tau\}}$ pointwise, implying that $Y\mathbf{1}_{\{\sigma\leqslant\tau\}}$ is $\mathcal{F}_{\sigma\wedge\tau}$-measurable, as required.

(b) We first write

$$E[Y \mid \mathcal{F}_\tau] = E[Y\mathbf{1}_{\{\sigma>\tau\}} \mid \mathcal{F}_\tau] + E[Y\mathbf{1}_{\{\sigma\leqslant\tau\}} \mid \mathcal{F}_\tau] = E[Y \mid \mathcal{F}_\tau]\mathbf{1}_{\{\sigma>\tau\}} + Y\mathbf{1}_{\{\sigma\leqslant\tau\}},$$

where in the last equality we use part (a) together with the fact that $\mathcal{F}_{\sigma\wedge\tau} \subseteq \mathcal{F}_\tau$. Part (a) implies that $E[Y \,|\, \mathcal{F}_\tau]\mathbf{1}_{\{\sigma>\tau\}} = E[Y \,|\, \mathcal{F}_\tau]\mathbf{1}_{\{\tau<\sigma\}}$ is $\mathcal{F}_{\sigma\wedge\tau}$-measurable. As Y is $\mathcal{F}_\sigma$-measurable, part (a) also gives that $Y\mathbf{1}_{\sigma\leqslant\tau}$ is $\mathcal{F}_{\sigma\wedge\tau}$-measurable, and thus $E[Y \,|\, \mathcal{F}_\tau]$ is also $\mathcal{F}_{\sigma\wedge\tau}$-measurable.

(c) Fix $Z \in L^1(\Omega)$. Then $E[Z \,|\, \mathcal{F}_\sigma]$ is $\mathcal{F}_\sigma$-measurable and integrable; so by part (b), $E[E[Z \,|\, \mathcal{F}_\sigma] \,|\, \mathcal{F}_\tau]$ is $\mathcal{F}_{\sigma\wedge\tau}$-measurable. Since $\mathcal{F}_{\sigma\wedge\tau} \subseteq \mathcal{F}_\tau$ and $\mathcal{F}_{\sigma\wedge\tau} \subseteq \mathcal{F}_\sigma$, we thus have

$$\begin{aligned} E\Big[E[Z \,|\, \mathcal{F}_\sigma] \,\Big|\, \mathcal{F}_\tau\Big] &= E\Big[E\Big[E[Z \,|\, \mathcal{F}_\sigma] \,\Big|\, \mathcal{F}_\tau\Big] \,\Big|\, \mathcal{F}_{\sigma\wedge\tau}\Big] \\ &= E\Big[E[Z \,|\, \mathcal{F}_\sigma] \,\Big|\, \mathcal{F}_{\sigma\wedge\tau}\Big] \\ &= E[Z \,|\, \mathcal{F}_{\sigma\wedge\tau}]. \end{aligned}$$

By symmetry, we therefore also have

$$E\Big[E[Z \,|\, \mathcal{F}_\tau] \,\Big|\, \mathcal{F}_\sigma\Big] = E[Z \,|\, \mathcal{F}_{\sigma\wedge\tau}],$$

which completes the proof.

Exercise 4.2 *(Stopped martingales)* Let $M = (M_t)_{t\geqslant 0}$ be a martingale with right-continuous sample paths and let τ be a stopping time with respect to the same filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$. Define the *stopped process* $M^\tau = (M^\tau_t)_{t\geqslant 0}$ by

$$M^\tau_t := M_{t\wedge\tau}.$$

(a) Suppose additionally that M is uniformly integrable. Show that for each $t \geqslant 0$,

$$M^\tau_t = E[M_\tau \,|\, \mathcal{F}_t].$$

Deduce that M^τ is a uniformly integrable martingale.

(b) Without assuming that M is uniformly integrable, show that the stopped process M^τ is still a martingale.

Solution 4.2

(a) Fix $t \geqslant 0$. We have that $t \wedge \tau$ is a stopping time with $t \wedge \tau \leqslant \tau$. Since M is a uniformly integrable martingale, we can apply the stopping theorem (Theorem 2.3.8) to get

$$E[M_\tau \,|\, \mathcal{F}_{t\wedge\tau}] = M_{t\wedge\tau}. \tag{1}$$

In particular, we see that $M_{t\wedge\tau}$ is integrable and $\mathcal{F}_{t\wedge\tau}$-measurable. Since $\mathcal{F}_{t\wedge\tau} \subseteq \mathcal{F}_t$ (by Exercise 3.4(a)), we have that $M_{t\wedge\tau}$ is also $\mathcal{F}_t$-measurable. Thus in order to prove $M_{t\wedge\tau} = E[M_\tau \,|\, \mathcal{F}_t]$, it suffices to show that for each $A \in \mathcal{F}_t$,

$$E[\mathbf{1}_A M_\tau] = E[\mathbf{1}_A M_{t\wedge\tau}],$$

by the definition of the conditional expectation. To this end, fix $A \in \mathcal{F}_t$ and write

$$E[\mathbf{1}_{A\cap\{\tau\leqslant t\}}M_\tau] = E[\mathbf{1}_{A\cap\{\tau\leqslant t\}}M_{t\wedge\tau}]. \tag{2}$$

Notice also that by taking $S \equiv t$ in Exercise 3.4(c) gives $A \cap \{\tau > t\} \in \mathcal{F}_{t\wedge\tau}$. This together with (1) implies that

$$E[\mathbf{1}_{A\cap\{\tau>t\}}M_\tau] = E[\mathbf{1}_{A\cap\{\tau>t\}}M_{t\wedge\tau}].$$

Summing with (2) then yields the required equality. We have thus shown that $M^\tau_t = E[M_\tau \,|\, \mathcal{F}_t]$. In particular, M^τ is a martingale closed on the right and thus uniformly integrable.

(b) Fix $n \in \mathbb{N}$ and consider the process $(M_{t\wedge n})_{t\geqslant 0}$. Notice that $M_{t\wedge n} = E[M_n \,|\, \mathcal{F}_t]$ for all $t \geqslant 0$, so that $(M_{t\wedge n})_{t\geqslant 0}$ is a closed and hence uniformly integrable martingale. We can then apply part (a) to deduce that the stopped process $(M_{t\wedge n\wedge\tau})_{t\geqslant 0}$ is a uniformly integrable martingale. In particular, it follows that M^τ is a martingale on $[0, n]$, as $(M^\tau_t)_{0\leqslant t\leqslant n} \equiv (M_{t\wedge n\wedge\tau})_{0\leqslant t\leqslant n}$. Now letting $n \to \infty$ gives that M^τ is a martingale on the whole of $[0,\infty)$, completing the proof.

Alternative solution: Fix $s \leqslant t$. Since $t \wedge \tau \leqslant t$ are bounded stopping times, the stopping theorem gives $M_{t\wedge\tau} = E[M_t \,|\, \mathcal{F}_{t\wedge\tau}]$. We can then apply Exercise 4.1(c) to get

$$E[M_{t\wedge\tau} \,|\, \mathcal{F}_s] = E\Big[E[M_t \,|\, \mathcal{F}_{t\wedge\tau}]\,\Big|\, \mathcal{F}_s\Big] = E[M_t \,|\, \mathcal{F}_{s\wedge\tau}].$$

Now noting that $s\wedge\tau \leqslant t$ are bounded stopping times, we can apply the stopping theorem again to get $E[M_t \,|\, \mathcal{F}_{s\wedge\tau}] = M_{s\wedge\tau}$. This completes the proof.

Exercise 4.3 *(Ruin problem for Brownian motion)* Let $W = (W_t)_{t\geqslant 0}$ be a Brownian motion. For each $x \in \mathbb{R}$, define the stopping time τ_x by

$$\tau_x := \inf\{t \geqslant 0 : W_t = x\}.$$

Fix $a < 0 < b$ and set $\tau := \tau_a \wedge \tau_b$.

(a) Show that for each $\lambda > 0$,

$$E[e^{-\lambda\tau}] = \frac{\cosh(\frac{b+a}{2}\sqrt{2\lambda})}{\cosh(\frac{b-a}{2}\sqrt{2\lambda})}.$$

Hint: For a suitable choice of $\alpha \in \mathbb{R}$, consider the process $M = (M_t)_{t\geqslant 0}$ given by

$$M_t := e^{\sqrt{2\lambda}(W_t-\alpha)-\lambda t} + e^{-\sqrt{2\lambda}(W_t-\alpha)-\lambda t}.$$

You may want to think about why M is a martingale.

(b) Show similarly that for every $\lambda > 0$,

$$E[e^{-\lambda\tau}\mathbf{1}_{\{\tau=\tau_a\}}] = \frac{\sinh(b\sqrt{2\lambda})}{\sinh((b-a)\sqrt{2\lambda})}.$$

(c) Find the value of $P[\tau_a < \tau_b]$.

Hint: You may use the identity

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y).$$

Solution 4.3

(a) By Proposition 2.3.4, the processes $U = (U_t)_{t\geqslant 0}$ and $V = (V_t)_{t\geqslant 0}$ given by

$$U_t := e^{\sqrt{2\lambda}W_t - \lambda t} \quad \text{and} \quad V_t := e^{-\sqrt{2\lambda}W_t - \lambda t}$$

are martingales. Also, by Exercise 4.2(b), the stopped processes U^τ and V^τ are martingales. Moreover, by the definition of τ, we have for all $t \geqslant 0$ that

$$0 < U_t^\tau \leqslant e^{\sqrt{2\lambda}b} \quad \text{and} \quad 0 < V_t^\tau \leqslant e^{-\sqrt{2\lambda}a}.$$

It follows that U^τ and V^τ are in fact uniformly integrable martingales.

Now choose $\alpha = \frac{b+a}{2}$ and consider the corresponding process M as in the hint. We can write

$$M_t = e^{-\sqrt{2\lambda}\alpha}U_t + e^{\sqrt{2\lambda}\alpha}V_t,$$

In particular, M^τ is a linear combination of the uniformly integrable martingales U^τ and V^τ and thus is also a uniformly integrable martingale. We can thus apply the stopping theorem with stopping times $0 \leqslant \tau$ to get

$$E[M_\tau] = E[M_0] = 2\cosh\left(\sqrt{2\lambda}\frac{b+a}{2}\right).$$

On the other hand, since τ_a can never equal τ_b, we have

$$\begin{aligned}
E[M_\tau] &= E[M_\tau \mathbf{1}_{\{\tau_a<\tau_b\}}] + E[M_\tau \mathbf{1}_{\{\tau_a>\tau_b\}}] \\
&= e^{-\sqrt{2\lambda}\frac{b-a}{2}}E[e^{-\lambda\tau}\mathbf{1}_{\{\tau_a<\tau_b\}}] + e^{\sqrt{2\lambda}\frac{b-a}{2}}E[e^{-\lambda\tau}\mathbf{1}_{\{\tau_a<\tau_b\}}] \\
&\quad + e^{\sqrt{2\lambda}\frac{b-a}{2}}E[e^{-\lambda\tau}\mathbf{1}_{\{\tau_a>\tau_b\}}] + e^{-\sqrt{2\lambda}\frac{b-a}{2}}E[e^{-\lambda\tau}\mathbf{1}_{\{\tau_a>\tau_b\}}] \\
&= (e^{\sqrt{2\lambda}\frac{b-a}{2}} + e^{-\sqrt{2\lambda}\frac{b-a}{2}})E[e^{-\lambda\tau}] \\
&= 2\cosh\left(\sqrt{2\lambda}\frac{b-a}{2}\right)E[e^{-\lambda\tau}].
\end{aligned}$$

It follows that

$$E[e^{-\lambda\tau}] = \frac{\cosh(\frac{b+a}{2}\sqrt{2\lambda})}{\cosh(\frac{b-a}{2}\sqrt{2\lambda})},$$

as required.

(b) Similarly to part (a), we consider the martingale $N = (N_t)_{t\geqslant 0}$ given by

$$N_t := e^{\sqrt{2\lambda}(W_t-\alpha)-\lambda t} - e^{-\sqrt{2\lambda}(W_t-\alpha)-\lambda t},$$

with $\alpha = \frac{b+a}{2}$. Arguing analogously as in part (a), we arrive at

$$E[N_\tau] = E[N_0] = -2\sinh\left(\sqrt{2\lambda}\frac{b+a}{2}\right).$$

On the other hand, we have

$$\begin{aligned} E[N_\tau] &= e^{-\sqrt{2\lambda}\frac{b-a}{2}} E[e^{-\lambda\tau}\mathbf{1}_{\{\tau_a<\tau_b\}}] - e^{\sqrt{2\lambda}\frac{b-a}{2}} E[e^{-\lambda\tau}\mathbf{1}_{\{\tau_a<\tau_b\}}] \\ &\quad + e^{\sqrt{2\lambda}\frac{b-a}{2}} E[e^{-\lambda\tau}\mathbf{1}_{\{\tau_a>\tau_b\}}] - e^{-\sqrt{2\lambda}\frac{b-a}{2}} E[e^{-\lambda\tau}\mathbf{1}_{\{\tau_a>\tau_b\}}] \\ &= -2\sinh\left(\sqrt{2\lambda}\frac{b-a}{2}\right) E[e^{-\lambda\tau}\mathbf{1}_{\{\tau_a<\tau_b\}}] \\ &\quad + 2\sinh\left(\sqrt{2\lambda}\frac{b-a}{2}\right) E[e^{-\lambda\tau}\mathbf{1}_{\{\tau_a>\tau_b\}}], \end{aligned}$$

so that

$$\begin{aligned} -2\sinh\left(\sqrt{2\lambda}\frac{b+a}{2}\right) &= -2\sinh\left(\sqrt{2\lambda}\frac{b-a}{2}\right) E[e^{-\lambda\tau}\mathbf{1}_{\{\tau_a<\tau_b\}}] \\ &\quad + 2\sinh\left(\sqrt{2\lambda}\frac{b-a}{2}\right) E[e^{-\lambda\tau}\mathbf{1}_{\{\tau_a>\tau_b\}}]. \end{aligned} \tag{3}$$

From part (a), we have

$$\begin{aligned} 2\cosh\left(\sqrt{2\lambda}\frac{b+a}{2}\right) &= E[M_\tau] = 2\cosh\left(\sqrt{2\lambda}\frac{b-a}{2}\right) E[e^{-\lambda\tau}] \\ &= 2\cosh\left(\sqrt{2\lambda}\frac{b-a}{2}\right) E[e^{-\lambda\tau}\mathbf{1}_{\{\tau_a<\tau_b\}}] \\ &\quad + 2\cosh\left(\sqrt{2\lambda}\frac{b-a}{2}\right) E[e^{-\lambda\tau}\mathbf{1}_{\{\tau_a>\tau_b\}}]. \end{aligned} \tag{4}$$

We now multiply (3) by $\cosh(\sqrt{2\lambda}\frac{b-a}{2})$ and (4) by $\sinh(\sqrt{2\lambda}\frac{b-a}{2})$ and subtract to get

$$\begin{aligned} &\cosh\left(\sqrt{2\lambda}\frac{b+a}{2}\right)\sinh\left(\sqrt{2\lambda}\frac{b-a}{2}\right) + \sinh\left(\sqrt{2\lambda}\frac{b+a}{2}\right)\cosh\left(\sqrt{2\lambda}\frac{b-a}{2}\right) \\ &= \cosh\left(\sqrt{2\lambda}\frac{b-a}{2}\right)\sinh\left(\sqrt{2\lambda}\frac{b-a}{2}\right) E[e^{-\lambda\tau}\mathbf{1}_{\{\tau_a<\tau_b\}}] \\ &\quad + \sinh\left(\sqrt{2\lambda}\frac{b-a}{2}\right)\cosh\left(\sqrt{2\lambda}\frac{b-a}{2}\right) E[e^{-\lambda\tau}\mathbf{1}_{\{\tau_a<\tau_b\}}]. \end{aligned}$$

Using the identity $\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$ twice, we rewrite the above as

$$\sinh(b\sqrt{2\lambda}) = \sinh\left((b-a)\sqrt{2\lambda}\right) E[e^{-\lambda\tau}\mathbf{1}_{\{\tau_a<\tau_b\}}].$$

Rearranging gives

$$E[e^{-\lambda\tau}\mathbf{1}_{\{\tau=\tau_a\}}] = E[e^{-\lambda\tau}\mathbf{1}_{\{\tau_a<\tau_b\}}] = \frac{\sinh(b\sqrt{2\lambda})}{\sinh((b-a)\sqrt{2\lambda})},$$

as required.

(c) We compute, using part (b) and the dominated convergence theorem,

$$\begin{aligned} P[\tau_a < \tau_b] &= E[\mathbf{1}_{\{\tau_a<\tau_b\}}] \\ &= \lim_{\lambda\downarrow 0} E[e^{-\lambda\tau}\mathbf{1}_{\{\tau_a<\tau_b\}}] \\ &= \lim_{\lambda\downarrow 0} \frac{\sinh(b\sqrt{2\lambda})}{\sinh((b-a)\sqrt{2\lambda})} \\ &= \lim_{\lambda\downarrow 0} \frac{\sinh(b\lambda)}{\sinh((b-a)\lambda)} \\ &= \frac{b}{b-a}\lim_{\lambda\downarrow 0} \frac{\cosh(b\lambda)}{\cosh((b-a)\lambda)} \\ &= \frac{b}{b-a}. \end{aligned}$$