Coordinator
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Brownian Motion and Stochastic Calculus

Exercise Sheet 5

Submit by 12:00 on Wednesday, March 26 via the course homepage.

Exercise 5.1 Let $(S, \mathcal{S})$ be a measurable space, $Y = (Y_t)_{t \geqslant 0}$ the canonical process on $(S^{[0,\infty)}, \mathcal{S}^{[0,\infty)})$, i.e., $Y_t(y) = y(t)$ for $y \in S^{[0,\infty)}$, $t \geqslant 0$, and $(K_t)_{t \geqslant 0}$ a transition semigroup on $(S, \mathcal{S})$. Moreover, for each $x \in S$, assume that there exists a unique probability measure $\mathbb{P}_x$ on $(S^{[0,\infty)}, \mathcal{S}^{[0,\infty)})$ under which Y is a Markov process with transition semigroup $(K_t)_{t \geqslant 0}$ and initial distribution $\nu = \delta_{\{x\}}$.

Suppose $Z \geqslant 0$ is an $\mathcal{S}^{[0,\infty)}$-measurable random variable on $S^{[0,\infty)}$. Use the monotone class theorem to prove that the map $x \mapsto \mathbb{E}_x[Z], x \in S$, is $\mathcal{S}$-measurable.

Exercise 5.2 Suppose $X = (X_t)_{t \geqslant 0}$ is a right-continuous process with stationary and independent increments null at 0. Show that for any finite stopping time τ, the process $X^{(\tau)} = (X_t^{(\tau)})_{t \geqslant 0}$ given by

$$X_t^{(\tau)} := X_{\tau + t} - X_\tau$$

is independent of $\mathcal{F}_\tau$.

Hint: For any $p \in \mathbb{N}$, $0 \leqslant t_1 < \ldots, t_p$ and bounded and measurable $F : \mathbb{R}^p \to \mathbb{R}$, the stationary increments of X imply that

$$E[F(X_{t_1}, \ldots, X_{t_p})] = E[F(X_{t_1+h} - X_h, \ldots, X_{t_p+h} - X_h)], \quad \forall h \geqslant 0.$$

Exercise 5.3 Let $W = (W_t)_{t \geqslant 0}$ be a Brownian motion and for each $a \geqslant 0$, define the stopping time

$$T_a := \inf\{t \geqslant 0 : W_t = a\}.$$

Show that the stochastic process $T = (T_a)_{a \geqslant 0}$ has stationary and independent increments, in the sense that, for every $0 \leqslant a \leqslant b$, $T_b - T_a$ is independent of $\sigma(T_c : 0 \leqslant c \leqslant a)$ and has the same distribution as T_{b-a}.

Exercise 5.4

(a) Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, P)$ be a filtered probability space with $\Omega = \{\omega_1, \ldots, \omega_k\}$ finite and $\mathcal{F} = 2^\Omega$.

Show that the $\mathbb{R}^k$-valued process

$$X_t = (P[\{\omega_1\} \,|\, \mathcal{F}_t], \ldots, P[\{\omega_k\} \,|\, \mathcal{F}_t])$$

is a Markov process.

(b) Let W be a Brownian motion. Which of the following processes X are Markov? Write down the corresponding transition kernels in those cases.

1. $X_t = |W_t|$ (reflected Brownian motion).
2. $X_t = \int_0^t W_u \, \mathrm{d}u$ (integrated Brownian motion).
3. $X_t = W_{\tau_a \wedge t}$, where $\tau_a = \inf\{t \geqslant 0 : W_t \geqslant a\}$ is the hitting time of $a > 0$.
4. $X_t = W_t^\tau$ for a random time $\tau \sim \mathrm{Exp}(1)$ independent of W.
5. $X_t = t - t \wedge \tau$, where $\tau \sim \mathrm{Exp}(1)$ is a random time.