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Brownian Motion and Stochastic Calculus

Exercise Sheet 5

Submit by 12:00 on Wednesday, March 26 via the course homepage.

Exercise 5.1 Let $(S, \mathcal{S})$ be a measurable space, $Y = (Y_t)_{t \geqslant 0}$ the canonical process on $(S^{[0,\infty)}, \mathcal{S}^{[0,\infty)})$, i.e., $Y_t(y) = y(t)$ for $y \in S^{[0,\infty)}$, $t \geqslant 0$, and $(K_t)_{t \geqslant 0}$ a transition semigroup on $(S, \mathcal{S})$. Moreover, for each $x \in S$, assume that there exists a unique probability measure $\mathbb{P}_x$ on $(S^{[0,\infty)}, \mathcal{S}^{[0,\infty)})$ under which Y is a Markov process with transition semigroup $(K_t)_{t \geqslant 0}$ and initial distribution $\nu = \delta_{\{x\}}$.

Suppose $Z \geqslant 0$ is an $\mathcal{S}^{[0,\infty)}$-measurable random variable on $S^{[0,\infty)}$. Use the monotone class theorem to prove that the map $x \mapsto \mathbb{E}_x[Z], x \in S$, is $\mathcal{S}$-measurable.

Solution 5.1 Let $\mathcal{H}$ denote the real vector space of all bounded, $\mathcal{S}^{[0,\infty)}$-measurable functions $Z : S^{[0,\infty)} \to \mathbb{R}$ such that the map $x \mapsto \mathbb{E}_x[Z]$, $x \in S$, is $\mathcal{S}$-measurable. Since pointwise limits of measurable functions are measurable, $\mathcal{H}$ is closed under monotone bounded convergence. The family

$$\mathcal{M} = \left\{ \prod_{k=0}^{n} f_k(Y_{t_k}) : n \in \mathbb{N}, 0 = t_0 < t_1 < \cdots < t_n, f_k : S \to \mathbb{R} \ \mathcal{S}\text{-measurable, bdd} \right\}$$

is closed under multiplication and $\sigma(\mathcal{M}) = \mathcal{S}^{[0,\infty)}$. Clearly $1 \in \mathcal{M}$, and thus it remains to show that $\mathcal{M} \subseteq \mathcal{H}$. Indeed, for an element $Z = \prod_{k=0}^{n} f_k(Y_{t_k})$ in $\mathcal{M}$, we have for all $x \in S$ that

$$\begin{aligned}
\mathbb{E}_x[Z] &= \int_S \delta_{\{x\}}(\mathrm{d}x_0) f_0(x_0) \int_S K_{t_1 - t_0}(x_0, \mathrm{d}x_1) f_1(x_1) \cdots \int_S K_{t_n - t_{n-1}}(x_{n-1}, \mathrm{d}x_n) f_n(x_n) \\
&= f_0(x) \int_S K_{t_1 - t_0}(x, \mathrm{d}x_1) f_1(x_1) \cdots \int_S K_{t_n - t_{n-1}}(x_{n-1}, \mathrm{d}x_n) f_n(x_n). \qquad (1)
\end{aligned}$$

Using measure-theoretic induction, we can see that $x \mapsto \int_S g(y) K(x, \mathrm{d}y)$, $x \in S$, is $\mathcal{S}$-measurable for any bounded, $\mathcal{S}$-measurable function $g : S \to \mathbb{R}$ and any stochastic kernel K on $(S, \mathcal{S})$. Now notice that we can rewrite (1) as

$$\mathbb{E}_x[Z] = f_0(x) \int_S g(x_1) K_{t_1 - t_0}(x, \mathrm{d}x_1),$$

where

$$g(x_1) := f_1(x_1) \int_S K_{t_2 - t_1}(x_1, \mathrm{d}x_2) f_2(x_2) \cdots \int_S K_{t_n - t_{n-1}}(x_{n-1}, \mathrm{d}x_n) f_n(x_n).$$

So by proceeding by induction on n, we may assume that g is measurable and thus so is $x \mapsto \mathbb{E}_x[Z]$. We conclude that $x \mapsto \mathbb{E}_x[Z]$ is $\mathcal{S}$-measurable for any $Z \in \mathcal{M}$.

We can now apply the monotone class theorem to get that $\mathcal{H}$ contains all bounded, $\mathcal{S}^{[0,\infty)}$-measurable Z. For a general $\mathcal{S}^{[0,\infty)}$-measurable $Z \geqslant 0$, the montone convergence theorem implies for each $x \in S$ that $\mathbb{E}_x[Z] = \lim_{n\to\infty} \mathbb{E}_x[Z \wedge n]$. Thus, as a pointwise limit of $\mathcal{S}$-measurable functions, $x \mapsto \mathbb{E}_x[Z]$ is $\mathcal{S}$-measurable. This completes the proof.

Exercise 5.2 Suppose $X = (X_t)_{t\geqslant 0}$ is a right-continuous process with stationary and independent increments null at 0. Show that for any finite stopping time τ, the process $X^{(\tau)} = (X_t^{(\tau)})_{t\geqslant 0}$ given by

$$X_t^{(\tau)} := X_{\tau+t} - X_\tau$$

is independent of $\mathcal{F}_\tau$.

Hint: For any $p \in \mathbb{N}$, $0 \leqslant t_1 < \ldots, t_p$ and bounded and measurable $F : \mathbb{R}^p \to \mathbb{R}$, the stationary increments of X imply that

$$E[F(X_{t_1}, \ldots, X_{t_p})] = E[F(X_{t_1+h} - X_h, \ldots, X_{t_p+h} - X_h)], \quad \forall h \geqslant 0.$$

Solution 5.2 We fix $A \in \mathcal{F}_\tau$, $0 \leqslant t_1 < \cdots < t_p$ and $F : \mathbb{R}^p \to \mathbb{R}_+$ bounded and continuous. Notice that is is enough to show that

$$E[\mathbf{1}_A F(X_{t_1}^{(\tau)}, \ldots, X_{t_p}^{(\tau)})] = P[A] E[F(X_{t_1}, \ldots, X_{t_p})]. \tag{2}$$

Indeed, taking $A = \Omega$ in (2) yields

$$E[F(X_{t_1}, \ldots, X_{t_p})] = E[F(X_{t_1}^{(\tau)}, \ldots, X_{t_p}^{(\tau)})].$$

Then substituting this into (2) for a general $A \in \mathcal{F}_\tau$ gives

$$E[\mathbf{1}_A F(X_{t_1}^{(\tau)}, \ldots, X_{t_p}^{(\tau)})] = P[A] E[F(X_{t_1}^{(\tau)}, \ldots, X_{t_p}^{(\tau)})],$$

which implies that the vector $(X_{t_1}^{\tau}, \ldots, X_{t_p}^{(\tau)})$ is independent of $\mathcal{F}_\tau$ for every choice of times $0 \leqslant t_1 < \cdots < t_p$. From this, it follows that the whole process $X^{(\tau)}$ is independent of $\mathcal{F}_\tau$.

So it remains to establish (2). For every $n \in \mathbb{N}$ and $t \geqslant 0$, we write $[t]_n$ for the smallest real number of the form $k2^{-n}$, with $k \in \mathbb{Z}_+$, greater than or equal to t. With this notation, we write $X^{([\tau]_n])} = (X_t^{([\tau]_n)})_{t\geqslant 0}$ with $X_t^{([\tau]_n)} : \Omega \to \mathbb{R}$ given by $X_t^{([\tau]_n)}(\omega) := X_{[\tau(\omega)]_n+t}(\omega) - X_{[\tau(\omega)]_n}(\omega)$. Since X is right-continuous and F is continuous, we have

$$F(X_{t_1}^{(\tau)}, \dots, X_{t_p}^{(\tau)}) = \lim_{n\to\infty} F(X_{t_1}^{([\tau]_n)}, \dots, X_{t_p}^{([\tau]_n)}).$$

Using that F is bounded, we can then apply the dominated convergence theorem to get

$$E[\mathbf{1}_A F(X_{t_1}^{(\tau)}, \dots, X_{t_p}^{(\tau)})] = \lim_{n\to\infty} E[\mathbf{1}_A F(X_{t_1}^{([\tau]_n)}, \dots, X_{t_p}^{([\tau]_n)})].$$

Now for fixed n, notice that $[\tau]_n$ takes values in $\{k2^{-n} : k \in \mathbb{Z}_+\}$, with $[\tau]_n(\omega) = k2^{-n}$ if and only if $(k-1)2^{-n} < \tau(\omega) \leqslant k2^{-n}$. So we can write

$$\begin{aligned}
&E[\mathbf{1}_A F(X_{t_1}^{([\tau]_n)}, \dots, X_{t_p}^{([\tau]_n)})] \\
&\quad = \sum_{k=0}^{\infty} E[\mathbf{1}_A \mathbf{1}_{\{(k-1)2^{-n} < \tau \leqslant k2^{-n}\}} F(X_{k2^{-n}+t_1} - X_{k2^{-n}}, \dots, X_{k2^{-n}+t_p} - X_{k2^{-n}})] \\
&\quad = \sum_{k=0}^{\infty} E[\mathbf{1}_{A\cap\{(k-1)2^{-n} < \tau \leqslant k2^{-n}\}} F(X_{k2^{-n}+t_1} - X_{k2^{-n}}, \dots, X_{k2^{-n}+t_p} - X_{k2^{-n}})].
\end{aligned}$$

But the vector $(X_{k2^{-n}+t_1} - X_{k2^{-n}}, \dots, X_{k2^{-n}+t_p} - X_{k2^{-n}})$ is independent of $\mathcal{F}_{k2^{-n}}$, since X has independent increments. Moreover, we have

$$A \cap \{(k-1)2^{-n} < \tau \leqslant k2^{-n}\} = (A \cap \{\tau \leqslant k2^{-n}\}) \cap \{\tau \leqslant (k-1)2^{-n}\}^c \in \mathcal{F}_{k2^{-n}}.$$

We thus have

$$\begin{aligned}
&E[\mathbf{1}_{A\cap\{(k-1)2^{-n} < \tau \leqslant k2^{-n}\}} F(X_{k2^{-n}+t_1} - X_{k2^{-n}}, \dots, X_{k2^{-n}+t_p} - X_{k2^{-n}})] \\
&= P[A \cap \{(k-1)2^{-n} < \tau \leqslant k2^{-n}\}] E[F(X_{k2^{-n}+t_1} - X_{k2^{-n}}, \dots, X_{k2^{-n}+t_p} - X_{k2^{-n}})] \\
&= P[A \cap \{(k-1)2^{-n} < \tau \leqslant k2^{-n}\}] E[F(X_{t_1}, \dots, X_{t_p})],
\end{aligned}$$

where the hint is used to get the last equality. Summing over $k \in \mathbb{Z}_+$ gives

$$E[\mathbf{1}_A F(X_{t_1}^{([\tau]_n)}, \dots, X_{t_p}^{([\tau]_n)})] = P[A] E[F(X_{t_1}, \dots, X_{t_p})],$$

and letting $n \to \infty$ yields (2), completing the proof.

Exercise 5.3 Let $W = (W_t)_{t\geqslant 0}$ be a Brownian motion and for each $a \geqslant 0$, define the stopping time

$$T_a := \inf\{t \geqslant 0 : W_t = a\}.$$

Show that the stochastic process $T = (T_a)_{a\geqslant 0}$ has stationary and independent increments, in the sense that, for every $0 \leqslant a \leqslant b$, $T_b - T_a$ is independent of $\sigma(T_c : 0 \leqslant c \leqslant a)$ and has the same distribution as T_{b-a}.

Solution 5.3 Fix $0 \leqslant a \leqslant b$. We first show that $T_b - T_a \overset{(d)}{=} T_{b-a}$. To this end, define the process $W^{(T_a)} = (W_t^{(T_a)})_{t \geqslant 0}$ by

$$W_t^{(T_a)} := \mathbf{1}_{\{T_a < \infty\}}(W_{T_a+t} - W_{T_a}).$$

Since $T_a < \infty$ P-a.s., $W^{(T_a)}$ is a Brownian motion (by Example 3.2.23 and Theorem 3.2.28, as discussed in the proof of Theorem 3.3.2). For each $c \in \mathbb{R}$, define the stopping time

$$S_c := \inf\{t \geqslant 0 : W_t^{(T_a)} = c\}.$$

Then as $W^{(T_a)}$ is a Brownian motion, we have $T_{b-a} \overset{(d)}{=} S_{b-a}$. On the other hand, we have

$$\begin{aligned} S_{b-a} &= \inf\{t \geqslant 0 : W_t^{(T_a)} = b - a\} \\ &= \inf\{t \geqslant 0 : W_{T_a+t} = b\} \\ &= \inf\{T_a + t : t \geqslant 0 \text{ and } W_{T_a+t} = b\} - T_a \\ &= \inf\{s : s \geqslant T_a \text{ and } W_s = b\} - T_a. \end{aligned}$$

Since $0 \leqslant a \leqslant b$, we have that (for almost every $\omega \in \Omega$) $W_s(\omega) = b$ only if $s \geqslant T_a(\omega)$. Therefore,

$$\begin{aligned} S_{b-a} &= \inf\{s : s \geqslant T_a \text{ and } W_s = b\} - T_a \\ &= \inf\{s \geqslant 0 : W_s = b\} - T_a \\ &= T_b - T_a. \end{aligned}$$

Hence $T_{b-a} \overset{(d)}{=} T_b - T_a$, as required.

It remains to show that $T_b - T_a$ is independent of $\sigma(T_c : 0 \leqslant c \leqslant a)$. We first prove that $T_b - T_a$ is independent of $\mathcal{F}_{T_a}$. Recalling that $S_{b-a} = T_b - T_a$, we write

$$\{S_{b-a} \leqslant t\} = \left\{ \inf_{s \in \mathbb{Q} \cap [0,t]} |W_s^{(T_a)} - (b-a)| = 0 \right\}.$$

By the strong Markov property of Brownian motion (or by Exercise 5.2), $W^{(T_a)}$ is independent of $\mathcal{F}_{T_a}$. The above equality implies that the event $\{S_{b-a} \leqslant t\}$ is also independent of $\mathcal{F}_{T_a}$, and thus so is S_{b-a}. We thus have that $T_b - T_a$ is independent of $\mathcal{F}_{T_a}$. It now suffices to show that $\sigma(T_c : 0 \leqslant c \leqslant a) \subseteq \mathcal{F}_{T_a}$.

For each $0 \leqslant c \leqslant a$ and $t \geqslant 0$, we can easily see that $\{T_c \leqslant t\} \in \mathcal{F}_{T_c}$. We also have $\mathcal{F}_{T_c} \subseteq \mathcal{F}_{T_a}$, and thus $\{T_c \leqslant t\} \in \mathcal{F}_{T_a}$. It follows that $\sigma(T_c) \subseteq \mathcal{F}_{T_a}$ for all $0 \leqslant c \leqslant a$, so that $\sigma(T_c : 0 \leqslant c \leqslant a) \subseteq \mathcal{F}_{T_a}$, completing the proof.

Exercise 5.4

(a) Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, P)$ be a filtered probability space with $\Omega = \{\omega_1, \dots, \omega_k\}$ finite and $\mathcal{F} = 2^\Omega$.

Show that the $\mathbb{R}^k$-valued process

$$X_t = (P[\{\omega_1\} \,|\, \mathcal{F}_t], \dots, P[\{\omega_k\} \,|\, \mathcal{F}_t])$$

is a Markov process.

(b) Let W be a Brownian motion. Which of the following processes X are Markov? Write down the corresponding transition kernels in those cases.

1. $X_t = |W_t|$ (reflected Brownian motion).
2. $X_t = \int_0^t W_u \,\mathrm{d}u$ (integrated Brownian motion).
3. $X_t = W_{\tau_a \wedge t}$, where $\tau_a = \inf\{t \geqslant 0 : W_t \geqslant a\}$ is the hitting time of $a > 0$.
4. $X_t = W_t^\tau$ for a random time $\tau \sim \mathrm{Exp}(1)$ independent of W.
5. $X_t = t - t \wedge \tau$, where $\tau \sim \mathrm{Exp}(1)$ is a random time.

Solution 5.4

(a) Let $s \leqslant t$ and let g be a bounded measurable function. For $\omega \in \Omega$, we have

$$E[g(X_t) \,|\, \mathcal{F}_s](\omega) = \sum_{i=1}^k g\Big(X_t(\omega_i)\Big) P[\{\omega_i\} \,|\, \mathcal{F}_s](\omega) = \sum_{i=1}^k g\Big(X_t(\omega_i)\Big) X_s^i(\omega).$$

Note that the $g(X_t(\omega_i))$ are constants and therefore the conditional expectation is a (linear) function of X_s; so the process is Markov.

(b) 1. This is a Markov process. Let $(\mathcal{F}_t^W)_{t\geqslant 0}, (\mathcal{F}_t^{|W|})_{t\geqslant 0}$ be the filtrations generated by $W, |W|$, respectively. For Borel $A \subseteq [0, \infty)$, $t \geqslant 0$ and $h > 0$, we have that

$$\begin{aligned} P[|W_{t+h}| \in A \,|\, \mathcal{F}_t^W] &= P[W_{t+h} \in A \,|\, \mathcal{F}_t^W] + P[-W_{t+h} \in A \,|\, \mathcal{F}_t^W] \\ &= \int_A \frac{1}{\sqrt{2\pi h}} \left(e^{-\frac{(y-W_t)^2}{2h}} + e^{-\frac{(y+W_t)^2}{2h}} \right) \mathrm{d}y \\ &= \int_A \frac{1}{\sqrt{2\pi h}} \left(e^{-\frac{(y-|W_t|)^2}{2h}} + e^{-\frac{(y+|W_t|)^2}{2h}} \right) \mathrm{d}y \\ &=: K_h(|W_t|, A). \end{aligned}$$

By the tower law and since the above is $\mathcal{F}_t^{|W|}$-measurable, we can write

$$P[|W_{t+h}| \in A \,|\, \mathcal{F}_t^{|W|}] = K_h(|W_t|, A) = P[|W_{t+h}| \in A \,|\, \sigma(|W_t|)],$$

so X is Markov.

2. This is not a Markov process. Let $(\mathcal{F}_t^X)$ be the filtration generated by X. For Borel $A \subseteq \mathbb{R}$,

$$\begin{aligned} P\left[X_t \in A \,\middle|\, \mathcal{F}_s^W\right] &= P\left[X_s + (t-s)W_s + \int_s^t (W_r - W_s)\,\mathrm{d}r \in A \,\middle|\, \mathcal{F}_s^W\right] \\ &= f_{t-s}\Big(X_s + (t-s)W_s, A\Big), \end{aligned}$$

where $f_t(x, A) = P[x + \int_0^t W_r\,\mathrm{d}r \in A]$, using the Markov property of W. We also note that $\mathcal{F}_t^W = \mathcal{F}_t^X$, where the inclusion "$\supseteq$" is immediate and "$\subseteq$" follows from $W_s = \lim_{\varepsilon\downarrow 0} \frac{X_s - X_{s-\varepsilon}}{\varepsilon}$. Therefore,

$$P[X_t \in A \,|\, \mathcal{F}_s^X] = P[X_t \in A \,|\, \mathcal{F}_s^W] = f_{t-s}\Big(X_s + (t-s)W_s, A\Big).$$

But $x \mapsto f_t(x, A)$ is injective (strictly increasing) for $A = [0, \infty)$ and W_s is not $\sigma(X_s)$-measurable, so X is not Markov.

3. This is a Markov process. Let $(\mathcal{F}_t^X)_{t\geqslant 0}$ be the filtration generated by X. For Borel $A \subseteq \mathbb{R}$, define $f_t^a(w, A) = P[w + W_{t\wedge\tau_{a-w}} \in A]$ for $t \geqslant 0$ and $0 \leqslant w \leqslant a$. Note that $\{\tau_a < t\} \in \mathcal{F}_t^W$ for all $t > 0$, and moreover

$$\{X_t = a\} = \{\tau_a \leqslant t\} = \{\tau_a < t\} \cup \{\tau_a = a\}.$$

Since $f_h^a(a, A) = \delta_a(A)$, we have

$$\begin{aligned} P[X_{t+h} \in A \,|\, \mathcal{F}_t^W] &= \mathbf{1}_{\{\tau_a<t\}}\delta_a(A) + f_h^a(W_t, A)\mathbf{1}_{\{\tau_a\geqslant t\}} \\ &= \mathbf{1}_{\{X_t=a\}}\delta_a(A) + f_h^a(X_t, A)\mathbf{1}_{\{X_t<a\}}, \end{aligned}$$

where the first line is justified by the Markov property of W and the second one follows from $\{\tau_a \leqslant t\} = \{X_t = a\}$. Since this is $\mathcal{F}_t^X$-measurable and $\mathcal{F}_t^X \subseteq \mathcal{F}_t^W$, we have

$$P[X_{t+h} \in A \,|\, \mathcal{F}_t^X] = P[X_{t+h} \in A \,|\, \mathcal{F}_t^W] = \mathbf{1}_{\{X_t=a\}}\delta_a(A) + f_h^a(X_t, A)\mathbf{1}_{\{X_t<a\}},$$

which is $\sigma(X_t)$-measurable, so X is Markov.

Remark: One can show that

$$f_t^a(w, (-\infty, y]) = \Phi\left(\frac{2a - y - w}{\sqrt{t}}\right) - \Phi\left(\frac{-y + w}{\sqrt{t}}\right),$$

for Φ the distribution function of a standard Gaussian and any $y < a$, while $f_t^a(w, \{a\}) = 2\Phi\Big(\frac{-a+w}{\sqrt{t}}\Big)$.

4. This is not a Markov process. Note that $\{\tau < t\} \in \mathcal{F}_t^X$ for each $t > 0$, since

$$\{\tau < t\} = \bigcup_{q\in(0,t)\cap\mathbb{Q}} \bigcap_{r\in(q,t)\cap\mathbb{Q}} \{X_r = X_q\} \in \mathcal{F}_t^X$$

as X stays constant after τ. Therefore,

$$P[X_t \in A \,|\, \mathcal{F}_s^X]\mathbf{1}_{\{\tau<s\}} = \delta_{X_s}(A)\mathbf{1}_{\{\tau<s\}}.$$

However, $\{\tau < s\} \notin \sigma(X_s)$; therefore X is not Markov.

5. This is a Markov process. Indeed, we have

$$\mathcal{F}_t^X = \sigma(\tau \wedge t) = \sigma(X_t),$$

since $\sigma(X_s) = \sigma(\tau \wedge s) = \sigma((\tau \wedge t) \wedge s) \subseteq \sigma(X_t)$ for $s \leqslant t$. It thus follows immediately that X is Markov. Note that on $\{X_t > 0\}$ we have $\tau < t$ and therefore $X_{t+h} = X_t + h$ P-a.s. On the other hand, on $\{X_t = 0\} = \{\tau \geqslant t\}$, we have $(\tau \,|\, \{\tau \geqslant t\}) \sim t + \mathrm{Exp}(1)$ by the memoryless property of the exponential distribution, and therefore $(X_{t+h} \,|\, X_t) \sim 0 \vee (h - \mathrm{Exp}(1))$. This allows us to compute the kernel

$$K_h(x, A) = \mathbf{1}_{\{x>0\}} \delta_{x+h}(A) + \mathbf{1}_{\{x=0\}} \left(e^{-h} \delta_{\{0\}}(A) + \int_0^h e^{-s} \mathbf{1}_{\{s \in A\}} \, \mathrm{d}s \right).$$