Coordinator
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Brownian Motion and Stochastic Calculus

Exercise Sheet 6

Submit by 12:00 on Wednesday, April 2 via the course homepage.

Exercise 6.1 Let $W = (W_t)_{t\geqslant 0}$ be a Brownian motion and define the stopping time $\sigma := \inf\{t \geqslant 0 : W_t > 0\}$. Prove that $P[\sigma = 0] = 1$, and for any $0 \leqslant a < b$ that

$$P\left[\max_{a\leqslant t\leqslant b} W_t > \max(W_a, W_b)\right] = 1. \tag{1}$$

Hint: As seen in the proof of Corollary 3.3.7, one can show that for any fixed $T > 0$, the process $(W_T - W_{T-t})_{0\leqslant t\leqslant T}$ is a Brownian motion on $[0, T]$.

Exercise 6.2 Let $(S, \mathcal{S}) = (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ and for each $x \in \mathbb{R}^2$, let $\mathbb{P}_x$ denote the unique probability measure on $(\mathbb{D}(S), \mathcal{D}(S))$ under which the coordinate process Y is a 2-dimensional Brownian motion starting at x. Show that Y is *neighbourhood recurrent* in the sense that for any $x \in \mathbb{R}^2$,

$$\mathbb{P}_x[\sup\{t \geqslant 0 : Y_t \in U\} = \infty \text{ for every non-empty open } \mathcal{O} \subseteq \mathbb{R}^2] = 1.$$

Hint: For any $z \in \mathbb{R}^2$ and $r > 0$, we denote by $B(z, r) := \{y \in \mathbb{R}^2 : |y - z| \leqslant r\}$ the closed ball of radius r centred at z. We also write $T_{B(z,r)} := \inf\{t \geqslant 0 : Y_t \in B(z, r)\}$. Use the fact that $T_{B(0,r)} < \infty$ $\mathbb{P}_x$-a.s. for any $x \in \mathbb{R}^2$ and $r > 0$,, and apply the strong Markov property of Brownian motion.

Exercise 6.3

(a) Let $(Z_t)_{t\geqslant 0}$ be an adapted process with respect to a given filtration $(\mathcal{F}_t)_{t\geqslant 0}$ and such that for every bounded continuous function $f : \mathbb{R} \to \mathbb{R}$, we have

$$E[f(Z_t - Z_s) \,|\, \mathcal{F}_s] = E[f(Z_{t-s})].$$

Show that Z has stationary independent increments.

(b) Let $(Z_t)_{t\geqslant 0}$ be a stochastic process null at zero with stationary independent increments. Let $(W_t)_{t\geqslant 0}$ be a Brownian motion independent of $(Z_t)_{t\geqslant 0}$ and let $(T_a)_{a\geqslant 0}$ be defined by

$$T_a := \inf\{s \geqslant 0 : W_s \geqslant a\}.$$

We know from Exercise 5.3 that the process $(T_a)_{a\geqslant 0}$ has stationary independent increments. Show that the process $(\widehat{Z}_t)_{t\geqslant 0} := (Z_{T_t})_{t\geqslant 0}$ also has stationary independent increments.

Remark: The process $(T_t)_{t\geqslant 0}$ is called a subordinator.

Exercise 6.4 For a right-continuous increasing function $f : [0,\infty) \to \mathbb{R}$, there exists a unique measure μ_f on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ such that $\mu_f(\{0\}) = f(0)$ and $\mu_f((0,t]) = f(t) - f(0)$ for all $t \geqslant 0$. We call a function $g : [0,\infty) \to \mathbb{R}$ *f-integrable* if $\int_{[0,\infty)} |g(s)|\, \mu_f(\mathrm{d}s) < \infty$. In this case, we define $\int g(s)\,\mathrm{d}f(s) := \int g(s)\,\mu_f(\mathrm{d}s)$.

In what follows, we let $f, g : [0,\infty) \to \mathbb{R}$ be right-continuous increasing functions.

(a) Assume that f is g-integrable. Show that the function $h(t) := \int_{[0,t]} f(s)\,\mathrm{d}g(s)$ is right-continuous. Moreover, show that if g is continuous, then h is continuous.

(b) Suppose f is g-integrable and g is f-integrable. Show the *integration-by-parts formula*: for each $t > 0$,

$$
\begin{aligned}
f(t)g(t) - f(0)g(0) &= \int_{(0,t]} f(s)\,\mathrm{d}g(s) + \int_{(0,t]} g(s-)\,\mathrm{d}f(s) \\
&= \int_{(0,t]} f(s-)\,\mathrm{d}g(s) + \int_{(0,t]} g(s)\,\mathrm{d}f(s).
\end{aligned}
$$

Remark: The above results still hold true if f and g are not increasing, but only of finite variation. Indeed, if f is right continuous and has finite variation, there exist increasing right-continuous functions $f_1, f_2 : [0,\infty) \to \mathbb{R}$ such that $f = f_1 - f_2$. We then say that g is f-integrable if the integrals $\int |g(s)|\,\mu_{f_1}(ds)$ and $\int |g(s)|\,\mu_{f_2}(ds)$ are both finite, and in this case, we define the integral $\int g(s)\,df(s) := \int g(s)\,df_1(s) - \int g(s)\,df_2(s)$. One can show that this integral is well defined in the sense that it is independent of the choice of functions f_1 and f_2.