Brownian Motion and Stochastic Calculus

Exercise Sheet 6

Submit by 12:00 on Wednesday, April 2 via the course homepage.

Exercise 6.1 Let $W = (W_t)_{t \geqslant 0}$ be a Brownian motion and define the stopping time $\sigma := \inf\{t \geqslant 0 : W_t > 0\}$. Prove that $P[\sigma = 0] = 1$, and for any $0 \leqslant a < b$ that

$$P\left[\max_{a \leqslant t \leqslant b} W_t > \max(W_a, W_b)\right] = 1. \tag{1}$$

Hint: As seen in the proof of Corollary 3.3.7, one can show that for any fixed $T > 0$, the process $(W_T - W_{T-t})_{0 \leqslant t \leqslant T}$ is a Brownian motion on $[0, T]$.

Solution 6.1 We prove that almost surely, for every rational $\varepsilon > 0$,

$$\max_{0 \leqslant t \leqslant \varepsilon} W_t > 0 \quad \text{and} \quad \min_{0 \leqslant t \leqslant \varepsilon} W_t < 0. \tag{2}$$

Once (2) is shown, we can see that on this probability-1 event, $\sigma \leqslant \varepsilon$ for each rational $\varepsilon > 0$, which yields $P[\sigma = 0] = 1$. In fact, it would already be enough to have only $P[\max_{0 \leqslant t \leqslant \varepsilon} W_t > 0 \text{ for every rational } \varepsilon > 0] = 1$, but we will need the analogous result for the minimum later.

To establish (2), we set

$$A := \left\{\max_{0 \leqslant t \leqslant \varepsilon} W_t > 0 \text{ for all rational } \varepsilon > 0\right\},$$

and notice that we can write, for each $k \in \mathbb{N}$,

$$A = \bigcap_{n=k}^{\infty} \left\{\max_{0 \leqslant t \leqslant 1/n} W_t > 0\right\}. \tag{3}$$

It follows that $A \in \mathcal{F}_{1/k}$ for each $k \in \mathbb{N}$, and hence $A \in \bigcap_{k=1}^{\infty} \mathcal{F}_{1/k} = \mathcal{F}_{0+}$. So by Blumenthal's 0-1 law, $P[A] \in \{0, 1\}$. On the other hand, since the intersection in (3) is decreasing, we have

$$P[A] = \lim_{n \to \infty} P\left[\max_{0 \leqslant t \leqslant 1/n} W_t > 0\right] \geqslant \liminf_{n \to \infty} P\left[W_{1/n} > 0\right] = \frac{1}{2},$$

and therefore $P[A] = 1$. To show the corresponding result for the minimum of Brownian motion, we just replace W with $-W$ in the above argument and recall that $-W$ is still a Brownian motion. This completes the proof of (2).

It remains to show (1). So fix $0 \leqslant a < b$ and note that

$$P\Big[\max_{a\leqslant t\leqslant b} W_t > W_a\Big] = P\Big[\max_{a\leqslant t\leqslant b}(W_t - W_a) > 0\Big] = P\Big[\max_{0\leqslant t\leqslant b-a}(W_{a+t} - W_a) > 0\Big].$$

By Proposition 2.1.1, $(W_{a+t} - W_a)_{t\geqslant 0}$ is a Brownian motion, and thus

$$P\Big[\max_{0\leqslant t\leqslant b-a}(W_{a+t} - W_a) > 0\Big] = P\Big[\max_{0\leqslant t\leqslant b-a} W_t > 0\Big] = 1,$$

by (2). We have thus shown that $P[\max_{a\leqslant t\leqslant b} W_t > W_a] = 1$. It now only remains to show that $P[\max_{a\leqslant t\leqslant b} W_t > W_b] = 1$. To this end, we use the hint to write

$$\begin{aligned} P\Big[\max_{a\leqslant t\leqslant b} W_t > W_b\Big] &= P\Big[\max_{a\leqslant t\leqslant b}(W_b - W_{b-t}) > W_b\Big] \\ &= P\Big[-\min_{a\leqslant t\leqslant b} W_{b-t} > 0\Big] \\ &= P\Big[\min_{0\leqslant t\leqslant b-a} W_t < 0\Big] \\ &= 1, \end{aligned}$$

by (2). This completes the proof.

Exercise 6.2 Let $(S, \mathcal{S}) = (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ and for each $x \in \mathbb{R}^2$, let $\mathbb{P}_x$ denote the unique probability measure on $(\mathbb{D}(S), \mathcal{D}(S))$ under which the coordinate process Y is a 2-dimensional Brownian motion starting at x. Show that Y is *neighbourhood recurrent* in the sense that for any $x \in \mathbb{R}^2$,

$$\mathbb{P}_x[\sup\{t \geqslant 0 : Y_t \in U\} = \infty \text{ for every non-empty open } \mathcal{O} \subseteq \mathbb{R}^2] = 1.$$

Hint: For any $z \in \mathbb{R}^2$ and $r > 0$, we denote by $B(z, r) := \{y \in \mathbb{R}^2 : |y - z| \leqslant r\}$ the closed ball of radius r centred at z. We also write $T_{B(z,r)} := \inf\{t \geqslant 0 : Y_t \in B(z, r)\}$. Use the fact that $T_{B(0,r)} < \infty$ $\mathbb{P}_x$-a.s. for any $x \in \mathbb{R}^2$ and $r > 0$,, and apply the strong Markov property of Brownian motion.

Solution 6.2 From the given hint, we know that

$$\text{for any } x \in \mathbb{R}^2 \text{ and } r > 0, \text{ we have } T_{B(0,r)} < \infty \ \ \mathbb{P}_x\text{-a.s.} \tag{4}$$

Moreover, $T_{B(0,r)}$ is a $\mathbb{Y}$-stopping time. Let $r > 0$. We define the sequence of $\mathbb{Y}$-stopping times $(T_i)_{i\in\mathbb{N}}$ by

$$\begin{aligned} T_1 &:= T_{B(0,r)}, \\ T_{i+1} &:= T_1 \circ \vartheta_{1+T_i} + 1 + T_i \\ &= \inf\{t \geqslant 1 + T_i : Y_t \in B(0, r)\} \quad \text{for } i \geqslant 1. \end{aligned}$$

Note that (T_i) increases strictly monotonically to infinity. For fixed $y \in \mathbb{R}^2$ and $i \in \mathbb{N}$, we compute

$$\begin{aligned}\mathbb{P}_y[T_{i+1} < \infty] &= \mathbb{E}_y[\mathbf{1}_{\{T_1 \circ \vartheta_{1+T_i} + 1 + T_i < \infty\}}] \\ &= \mathbb{E}_y[(\mathbf{1}_{\{T_1 < \infty\}} \circ \vartheta_{1+T_i})\, \mathbf{1}_{\{T_i < \infty\}}] \\ &= \mathbb{E}_y\Big[\mathbf{1}_{\{T_i < \infty\}}\, \mathbb{E}_y[\mathbf{1}_{\{T_1 < \infty\}} \circ \vartheta_{1+T_i} \,|\, \mathcal{Y}_{1+T_i}]\Big].\end{aligned}$$

By the strong Markov property, $\mathbb{E}_y[\mathbf{1}_{\{T_1 < \infty\}} \circ \vartheta_{1+T_i} \mid \mathcal{Y}_{1+T_i}] = \mathbb{E}_{Y_{1+T_i}}[\mathbf{1}_{\{T_1 < \infty\}}]$. So we can write

$$\begin{aligned}\mathbb{P}_y[T_{i+1} < \infty] &= \mathbb{E}_y\Big[\mathbf{1}_{\{T_i < \infty\}}\, \mathbb{E}_{Y_{1+T_i}}[\mathbf{1}_{\{T_1 < \infty\}}]\Big] \\ &= \mathbb{E}_y\Big[\mathbf{1}_{\{T_i < \infty\}}\, \mathbb{P}_{Y_{1+T_i}}[T_1 < \infty]\Big] \\ &= \mathbb{P}_y[T_i < \infty],\end{aligned}$$

where the in the last equality we use (4). It follows that $\mathbb{P}_y[T_i < \infty]$ is constant for all $i \in \mathbb{N}$, and thus $\mathbb{P}_y[T_i < \infty] = \mathbb{P}_y[T_1 < \infty] = 1$. Since Brownian motion has continuous sample paths almost surely and $B(0,r)$ is a closed set, we have that $Y_{T_i} \in B(0,r)$ $\mathbb{P}_y$-a.s. on $\{T_i < \infty\}$ for all $i \in \mathbb{N}$. It thus follows that for any $y \in \mathbb{R}^2$, the set $\{t \geqslant 0 : Y_t \in B(0, \frac{1}{n})\}$ is unbounded $\mathbb{P}_y$-a.s. for each $n \in \mathbb{N}$. Now, since $\mathbb{P}_y$ is the law of $(y + Y_t)_{t \geqslant 0}$ under $\mathbb{P}_0$, this implies that

$$\mathbb{P}_0\text{-a.s., the set } \{t \geqslant 0 : Y_t \in B(z, \tfrac{1}{n})\} \text{ is unbounded for all } z \in \mathbb{Q}^2, n \in \mathbb{N}.$$

This proves the claim for $x = 0$, as for every open set $\mathcal{O} \subseteq \mathbb{R}^2$, there exist $z \in \mathbb{Q}^2$ and $n \in \mathbb{N}$ with $B(z, \frac{1}{n}) \subseteq \mathcal{O}$. The case for general $x \in \mathbb{R}^2$ follows immediately since $(Y_t)_{t \geqslant 0}$ under $\mathbb{P}_x$ has the same law as $(x + Y_t)_{t \geqslant 0}$ under $\mathbb{P}_0$ and $\mathcal{O} - x \subseteq \mathbb{R}^2$ is open whenever $\mathcal{O}$ is open. This completes the proof.

Exercise 6.3

(a) Let $(Z_t)_{t \geqslant 0}$ be an adapted process with respect to a given filtration $(\mathcal{F}_t)_{t \geqslant 0}$ and such that for every bounded continuous function $f : \mathbb{R} \to \mathbb{R}$, we have

$$E[f(Z_t - Z_s) \,|\, \mathcal{F}_s] = E[f(Z_{t-s})].$$

Show that Z has stationary independent increments.

(b) Let $(Z_t)_{t \geqslant 0}$ be a stochastic process null at zero with stationary independent increments. Let $(W_t)_{t \geqslant 0}$ be a Brownian motion independent of $(Z_t)_{t \geqslant 0}$ and let $(T_a)_{a \geqslant 0}$ be defined by

$$T_a := \inf\{s \geqslant 0 : W_s \geqslant a\}.$$

We know from Exercise 5.3 that the process $(T_a)_{a \geqslant 0}$ has stationary independent increments. Show that the process $(\widehat{Z}_t)_{t \geqslant 0} := (Z_{T_t})_{t \geqslant 0}$ also has stationary independent increments.

Remark: The process $(T_t)_{t \geqslant 0}$ is called a subordinator.

Solution 6.3

(a) To see that Z has independent increments, it suffices to show that for any times $0 \leqslant t_0 < t_1 < \cdots < t_n$ and measurable bounded functionals $f^i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, n$, we have

$$E\left[\prod_{i=1}^{n} f^i(Z_{t_i} - Z_{t_{i-1}})\right] = \prod_{i=1}^{n} E[f^i(Z_{t_i - t_{i-1}})]. \tag{5}$$

Indeed, taking $n = 1$ in (5) gives $E[f(Z_b - Z_a)] = E[f(Z_{b-a})]$ for each measurable bounded $f : \mathbb{R} \to \mathbb{R}$ and $0 \leqslant a < b$. Then using this equality in the right hand side of (5) for each i gives $E[\prod_{i=1}^{n} f^i(Z_{t_i} - Z_{t_{i-1}})] = \prod_{i=1}^{n} E[f^i(Z_{t_i} - Z_{t_{i-1}})]$, which shows that Z had independent increments.

To establish (5), notice first that if the f^i are bounded and continuous, we have by the assumption in the problem that

$$\begin{aligned} E\left[\prod_{i=1}^{n} f^i(Z_{t_i} - Z_{t_{i-1}})\right] &= E\left[E\left[\prod_{i=1}^{n} f^i(Z_{t_i} - Z_{t_{i-1}}) \,\middle|\, \mathcal{F}_{t_{n-1}}\right]\right] \\ &= E\left[\prod_{i=1}^{n-1} f^i(Z_{t_i} - Z_{t_{i-1}})\right] E[f^n(Z_{t_n - t_{n-1}})] \\ &= \prod_{i=1}^{n} E[f^i(Z_{t_i - t_{i-1}})], \end{aligned}$$

where we use induction on n to get the last equality. Now if for $i = 1, \ldots, n$, $A_i \in \mathcal{B}(\mathbb{R})$, we can find a sequence of continuous functions $(f^{i,m})_{m \in \mathbb{N}}$ with a uniform bound $\sup_{m,i \in \mathbb{N}} \operatorname{ess\,sup} f^{i,m} \leqslant 1$ such that $f^{i,m} \to \mathbf{1}_{A_i}$ pointwise on $\mathbb{R}$. Therefore, we find by two applications of the dominated convergence theorem that

$$\begin{aligned} E\left[\prod_{i=1}^{n} \mathbf{1}_{A_i}(Z_{t_i} - Z_{t_{i-1}})\right] &= \lim_{m \to \infty} E\left[\prod_{i=1}^{n} f^{i,m}(Z_{t_i} - Z_{t_{i-1}})\right] \\ &= \lim_{m \to \infty} \prod_{i=1}^{n} E[f^{i,m}(Z_{t_i - t_{i-1}})] = \prod_{i=1}^{n} E[\mathbf{1}_{A_i}(Z_{t_i - t_{i-1}})]. \end{aligned}$$

This extends to simple functions by linearity of the expectation, and then to measurable bounded functions by approximation with simple functions. We have thus proved (5) and hence Z has independent increments.

It remains to prove that Z has stationary increments, for which it suffices to show that for any $k \in \mathbb{N}$, $h \geqslant 0$ and bounded measurable $f : \mathbb{R}^k \to \mathbb{R}$,

$$E[f(Z_{t_1} - Z_{t_0}, \ldots, Z_{t_n} - Z_{t_{n-1}})] = E[f(Z_{t_1+h} - Z_{t_0+h}, \ldots, Z_{t_n+h} - Z_{t_{n-1}+h})].$$

Letting $\mathcal{H}$ denote the set of bounded measurable functions such that this is satisfied for all $h \geqslant 0$, we see that $\mathcal{H}$ is a real vector space, contains 1 and is closed under bounded monotone convergence by the dominated convergence theorem. Moreover, by (5), we have $\mathcal{H} \supseteq \mathcal{M}$, where $\mathcal{M}$ is the set of functions of the form $f(z_1, \ldots, z_n) = \prod_{i=1}^n f^i(z_i)$ for bounded measurable f^i, since

$$E\left[\prod_{i=1}^n f^i(Z_{t_i} - Z_{t_{i-1}})\right] = \prod_{i=1}^n E[f^i(Z_{t_i - t_{i-1}})] = E\left[\prod_{i=1}^n f^i(Z_{t_i+h} - Z_{t_{i-1}+h})\right]$$

for all $h \geqslant 0$. Using the monotone class theorem, we conclude that $\mathcal{H}$ contains all bounded measurable functions and therefore Z has stationary increments, completing the proof.

Remark: We also could have proved that Z has independent increments by using the monotone class theorem.

(b) Let $0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n$ and $f^i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, n$, be bounded and measurable functions. Using the independence properties together with part (a) and the fact that $(T_t)_{t\geqslant 0}$ has stationary independent increments, we obtain

$$\begin{aligned}
E\left[\prod_{i=1}^n f^i(\widehat{Z}_{t_i} - \widehat{Z}_{t_{i-1}})\right] &= E\left[\prod_{i=1}^n f^i(Z_{T_{t_i}} - Z_{T_{t_{i-1}}})\right] \\
&= E\left[E\left[\prod_{i=1}^n f^i(Z_{T_{t_i}} - Z_{T_{t_{i-1}}}) \,\middle|\, \mathcal{F}_\infty^W\right]\right] \\
&= E\left[E\left[\prod_{i=1}^n f^i(Z_{s_i} - Z_{s_{i-1}})\right]\Bigg|_{s_i = T_{t_i},\, i=1,\ldots,n}\right] \\
&= E\left[\left(\prod_{i=1}^n E[f^i(Z_{s_i} - Z_{s_{i-1}})]\right)\Bigg|_{s_i = T_{t_i},\, i=1,\ldots,n}\right] \\
&= E\left[\left(\prod_{i=1}^n E[f^i(Z_{s_i - s_{i-1}})]\right)\Bigg|_{s_i = T_{t_i},\, i=1,\ldots,n}\right].
\end{aligned}$$

Now for each $i = 1, \ldots, n$, define the deterministic function $g^i : \mathbb{R} \to \mathbb{R}$ by $g^i(x) = E[f^i(Z_x)]$. We can rewrite the above as

$$\begin{aligned}
E\left[\prod_{i=1}^n f^i(\widehat{Z}_{t_i} - \widehat{Z}_{t_{i-1}})\right] &= E\left[\left(\prod_{i=1}^n g^i(s_i - s_{i-1})\right)\Bigg|_{s_i = T_{t_i},\, i=1,\ldots,n}\right] \\
&= E\left[\prod_{i=1}^n g^i(T_{t_i} - T_{t_{i-1}})\right] \\
&= \prod_{i=1}^n E[g^i(T_{t_i} - T_{t_{i-1}})] \\
&= \prod_{i=1}^n E[g^i(T_{t_i - t_{i-1}})],
\end{aligned}$$

since T has stationary independent increments. Now by independence of Z and W, we have for each $i = 1, \ldots, n$, that $g^i(T_{t_i - t_{i-1}}) = E[f^i(Z_{T_{t_i - t_{i-1}}}) \mid \mathcal{F}^W_\infty]$ and thus $E[g^i(T_{t_i - t_{i-1}})] = E[f^i(Z_{T_{t_i - t_{i-1}}})]$. So we write

$$\begin{aligned} E\left[\prod_{i=1}^n f^i(\widehat{Z}_{t_i} - \widehat{Z}_{t_{i-1}})\right] &= \prod_{i=1}^n E[f^i(Z_{T_{t_i - t_{i-1}}})] \\ &= \prod_{i=1}^n E[f^i(\widehat{Z}_{t_i - t_{i-1}})]. \end{aligned}$$

By the same argument as in part (a), this implies that $\widehat{Z}$ has independent and stationary increments, as required.

Exercise 6.4 For a right-continuous increasing function $f : [0, \infty) \to \mathbb{R}$, there exists a unique measure μ_f on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ such that $\mu_f(\{0\}) = f(0)$ and $\mu_f((0, t]) = f(t) - f(0)$ for all $t \geqslant 0$. We call a function $g : [0, \infty) \to \mathbb{R}$ *f-integrable* if $\int_{[0,\infty)} |g(s)| \, \mu_f(\mathrm{d}s) < \infty$. In this case, we define $\int g(s) \, \mathrm{d}f(s) := \int g(s) \, \mu_f(\mathrm{d}s)$.

In what follows, we let $f, g : [0, \infty) \to \mathbb{R}$ be right-continuous increasing functions.

(a) Assume that f is g-integrable. Show that the function $h(t) := \int_{[0,t]} f(s) \, \mathrm{d}g(s)$ is right-continuous. Moreover, show that if g is continuous, then h is continuous.

(b) Suppose f is g-integrable and g is f-integrable. Show the *integration-by-parts formula*: for each $t > 0$,

$$\begin{aligned} f(t)g(t) - f(0)g(0) &= \int_{(0,t]} f(s) \, \mathrm{d}g(s) + \int_{(0,t]} g(s-) \, \mathrm{d}f(s) \\ &= \int_{(0,t]} f(s-) \, \mathrm{d}g(s) + \int_{(0,t]} g(s) \, \mathrm{d}f(s). \end{aligned}$$

Remark: The above results still hold true if f and g are not increasing, but only of finite variation. Indeed, if f is right continuous and has finite variation, there exist increasing right-continuous functions $f_1, f_2 : [0, \infty) \to \mathbb{R}$ such that $f = f_1 - f_2$. We then say that g is f-integrable if the integrals $\int |g(s)| \, \mu_{f_1}(ds)$ and $\int |g(s)| \, \mu_{f_2}(ds)$ are both finite, and in this case, we define the integral $\int g(s) \, df(s) := \int g(s) \, df_1(s) - \int g(s) \, df_2(s)$. One can show that this integral is well defined in the sense that it is independent of the choice of functions f_1 and f_2.

Solution 6.4

(a) Fix $t \geqslant 0$ and let $(t_n)_{n \in \mathbb{N}} \subseteq [0, \infty)$ be a sequence which decreases to t. By the dominated convergence theorem, we have

$$\begin{aligned} h(t) &= \int_{[0,t]} f(s)\,\mathrm{d}g(s) \\ &= \int_{[0,\infty)} \mathbf{1}_{[0,t]}(s) f(s)\,\mu_g(\mathrm{d}s) \\ &= \lim_{n\to\infty} \int_{[0,\infty)} \mathbf{1}_{[0,t_n]}(s) f(s)\,\mu_g(\mathrm{d}s) \\ &= \lim_{n\to\infty} \int_{[0,t_n]} f(s)\,\mathrm{d}g(s) \\ &= \lim_{n\to\infty} h(t_n), \end{aligned}$$

which proves that h is right-continuous. Now assume that g is continuous. Fix $t > 0$ and let $(t_n)_{n\in\mathbb{N}} \subseteq [0,\infty)$ be a sequence which strictly increases to t. Again by the dominated convergence theorem, we have

$$\begin{aligned} \lim_{n\to\infty} h(t_n) &= \lim_{n\to\infty} \int_{[0,t_n]} f(s)\,\mathrm{d}g(s) \\ &= \lim_{n\to\infty} \int_{[0,\infty)} \mathbf{1}_{[0,t_n]}(s) f(s)\,\mu_g(\mathrm{d}s) \\ &= \int_{[0,\infty)} \mathbf{1}_{[0,t)}(s) f(s)\,\mu_g(\mathrm{d}s) \\ &= \int_{[0,\infty)} \mathbf{1}_{[0,t]}(s) f(s)\,\mu_g(\mathrm{d}s) - \int_{\{t\}} f(s)\,\mu_g(\mathrm{d}s) \\ &= \int_{[0,t]} f(s)\,\mathrm{d}g(s) - \int_{\{t\}} f(s)\,\mu_g(\mathrm{d}s) \\ &= h(t) - \int_{\{t\}} f(s)\,\mu_g(\mathrm{d}s). \end{aligned}$$

It thus remains to show that $\int_{\{t\}} f(s)\,\mu_g(\mathrm{d}s) = 0$. As $\mu_g((0,t]) = g(t) - g(0)$, we have that $\mu_g(\{t\}) = g(t) - g(t-) = 0$ since g is continuous. Therefore, we indeed have $\int_{\{t\}} f(s)\,\mu_g(\mathrm{d}s) = 0$, completing the proof.

(b) Fix $t \geqslant 0$. From Fubini's theorem, we obtain

$$\big(f(t)-f(0)\big)\big(g(t)-g(0)\big) = \int_{(0,t]} \mu_f(\mathrm{d}r) \int_{(0,t]} \mu_g(\mathrm{d}s) = \iint_{(0,t]\times(0,t]} \mu_f(\mathrm{d}r)\,\mu_g(\mathrm{d}s).$$

On the other hand, defining the domains

$$\begin{aligned} D_1 &:= \{(r,s) \in \mathbb{R}^2 : 0 < r \leqslant s \leqslant t\}, \\ D_2 &:= \{(r,s) \in \mathbb{R}^2 : 0 < s < r \leqslant t\}, \end{aligned}$$

we compute

$$
\begin{aligned}
\iint_{(0,t]\times(0,t]} \mu_f(\mathrm{d}r)\,\mu_g(\mathrm{d}s) &= \iint_{D_1} \mu_f(\mathrm{d}r)\,\mu_g(\mathrm{d}s) + \iint_{D_2} \mu_f(\mathrm{d}r)\,\mu_g(\mathrm{d}s) \\
&= \int_{(0,t]} \int_{(0,s]} \mu_f(\mathrm{d}r)\,\mu_g(\mathrm{d}s) + \int_{(0,t]} \int_{(0,r)} \mu_g(\mathrm{d}s)\,\mu_f(\mathrm{d}r) \\
&= \int_{(0,t]} \big(f(s) - f(0)\big)\,\mu_g(\mathrm{d}s) \\
&\quad + \int_{(0,t]} \big(g(r-) - g(0)\big)\,\mu_f(\mathrm{d}r) \\
&= \int_{(0,t]} f(s)\,\mathrm{d}g(s) - f(0)g(t) + f(0)g(0) \\
&\quad + \int_{(0,t]} g(s-)\,\mathrm{d}f(s) - g(0)f(t) + g(0)f(0).
\end{aligned}
$$

Comparing the above two expressions for $\iint_{(0,t]\times(0,t]} \mu_f(\mathrm{d}r)\,\mu_g(\mathrm{d}s)$ yields the result. The second equality follows by symmetry.