Brownian Motion and Stochastic Calculus

Exercise Sheet 7

Submit by 12:00 on Wednesday, April 9 via the course homepage.

Exercise 7.1 Let $M \in \mathcal{M}_{0,\mathrm{loc}}$. Establish the following properties.

(a) There exists a localising sequence $(\tau_n)_{n\in\mathbb{N}}$ for M such that for each $n \in \mathbb{N}$, the stopped process M^{τ_n} is a uniformly integrable martingale.

(b) If τ is a stopping time, then $M^\tau \in \mathcal{M}_{0,\mathrm{loc}}$.

(c) Let $(\tau_n)_{n\in\mathbb{N}}$ be a localising sequence for M and $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of stopping times with $\sigma_n \uparrow \infty$ P-a.s. Then $(\tau_n \wedge \sigma_n)_{n\in\mathbb{N}}$ is also a localising sequence for M.

(d) The space $\mathcal{M}_{0,\mathrm{loc}}$ is a vector space.

Exercise 7.2 Suppose that $M \in \mathcal{M}_{0,\mathrm{loc}}$ with $[M] \equiv 0$. Show that $M \equiv 0$ in the sense that M is indistinguishable from the 0 process.

Exercise 7.3 Let $M \in \mathcal{H}^2_0$. Show that $b\mathcal{E}$ is dense in $L^2(M)$.

Hint: Equip $\overline{\Omega} = \Omega \times [0,\infty)$ with the predictable σ-algebra $\mathcal{P}$. Let $C := E[M^2_\infty]$ and consider the probability measure $P_M = C^{-1} P \otimes [M]$ on $(\overline{\Omega}, \mathcal{P})$. Let $(\Pi_n)_{n\in\mathbb{N}}$ be an increasing sequence of partitions of $[0,\infty)$ with $\lim_{n\to\infty} |\Pi_n| = 0$. Use the martingale convergence theorem on $(\overline{\Omega}, \mathcal{P}, P_M)$ with respect to the discrete filtration $(\mathcal{P}_n)_{n\in\mathbb{N}}$ defined by

$$\mathcal{P}_n := \sigma(\{A_i \times (t_i, t_{i+1}] : t_i \in \Pi_n, A_i \in \mathcal{F}_{t_i}\}).$$

Exercise 7.4 For $M \in \mathcal{M}^c_{0,\mathrm{loc}}$, we denote by $L^2_{\mathrm{loc}}(M)$ the space of all predictable processes for which there exists a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ such that $\tau_n \uparrow \infty$ P-a.s. and $E[\int_0^{\tau_n} H_s^2 \,\mathrm{d}\langle M\rangle_s] < \infty$ for each $n \in \mathbb{N}$.

(a) Let H be predictable. Show that

$$H \in L^2_{\mathrm{loc}}(M) \iff \int_0^t H_s^2 \,\mathrm{d}\langle M\rangle_s < \infty \ P\text{-a.s. for each } t \geqslant 0.$$

(b) Show that for any continuous semimartingale X, any adapted RCLL process H and any sequence of partitions $(\Pi_n)_{n\in\mathbb{N}}$ of $[0,\infty)$ with $\lim_{n\to\infty} |\Pi_n| = 0$, we have

$$\int_0^{\cdot} H_{s-}\,\mathrm{d}X_s = \lim_{n\to\infty} \sum_{t_i\in\Pi_n} H_{t_i}\left(X_{t_{i+1}\wedge\cdot} - X_{t_i\wedge\cdot}\right) \quad \text{ucp},$$

where ucp stands for *uniformly on compacts in probability.*

(c) Find an adapted process with right-continuous paths which is not locally bounded.