Brownian Motion and Stochastic Calculus

Exercise Sheet 7

Submit by 12:00 on Wednesday, April 9 via the course homepage.

Exercise 7.1 Let $M \in \mathcal{M}_{0,\mathrm{loc}}$. Establish the following properties.

(a) There exists a localising sequence $(\tau_n)_{n\in\mathbb{N}}$ for M such that for each $n \in \mathbb{N}$, the stopped process M^{τ_n} is a uniformly integrable martingale.

(b) If τ is a stopping time, then $M^\tau \in \mathcal{M}_{0,\mathrm{loc}}$.

(c) Let $(\tau_n)_{n\in\mathbb{N}}$ be a localising sequence for M and $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of stopping times with $\sigma_n \uparrow \infty$ P-a.s. Then $(\tau_n \wedge \sigma_n)_{n\in\mathbb{N}}$ is also a localising sequence for M.

(d) The space $\mathcal{M}_{0,\mathrm{loc}}$ is a vector space.

Solution 7.1

(a) Let (T_n) be a localising sequence for M and for each $n \in \mathbb{N}$, define $\tau_n := T_n \wedge n$. Then $(\tau_n)_{n\in\mathbb{N}}$ is a sequence of stopping times with $\tau_n \uparrow \infty$ P-a.s. Now for each $n \in \mathbb{N}$, we can write $M^{\tau_n} = (M^{T_n}_{n\wedge t})_{t\geqslant 0}$. Since M^{T_n} is a martingale, it follows that M^{τ_n} is martingale closed on the right and thus uniformly integrable, as required.

(b) Let $(\tau_n)_{n\in\mathbb{N}}$ be a localising sequence for M. So for each $n \in \mathbb{N}$, M^{τ_n} is a martingale. Then by Exercise 4.2(b), we have that $(M^{\tau_n})^\tau$ is also a martingale. But $(M^{\tau_n})^\tau = (M^\tau)^{\tau_n}$, which shows that M^τ is indeed a local martingale with localising sequence $(\tau_n)_{n\in\mathbb{N}}$.

(c) Note that $(\tau_n \wedge \sigma_n)_{n\in\mathbb{N}}$ is a sequence of stopping times with $\tau_n \wedge \sigma_n \uparrow \infty$ P-a.s. So by the same reasoning as in part (b), we have that $M^{\tau_n\wedge\sigma_n} = (M^{\tau_n})^{\sigma_n}$ is a martingale for each $n \in \mathbb{N}$, and thus $(\tau_n \wedge \sigma_n)_{n\in\mathbb{N}}$ is a localising sequence for M.

(d) Let $(\tau_n)_{n\in\mathbb{N}}$ be a localising sequence for M. For $\lambda \in \mathbb{R}$, λM^{τ_n} is still a martingale for each $n \in \mathbb{N}$ by linearity of the conditional expectation, and thus $\lambda M \in \mathcal{M}_{0,\mathrm{loc}}$ (with localising sequence $(\tau_n)_{n\in\mathbb{N}}$). Now take $N \in \mathcal{M}_{0,\mathrm{loc}}$ with localising sequence $(\sigma_n)_{n\in\mathbb{N}}$. By part (c), $M^{\tau_n\wedge\sigma_n}$ and $N^{\tau_n\wedge\sigma_n}$ are martingales for each $n \in \mathbb{N}$, and thus so is $M^{\tau_n\wedge\sigma_n} + N^{\tau_n\wedge\sigma_n} = (M+N)^{\tau_n\wedge\sigma_n}$. It follows that $M + N \in \mathcal{M}_{0,\text{loc}}$ with localising sequence $(\tau_n \wedge \sigma_n)_{n\in\mathbb{N}}$. As $0 \in \mathcal{M}_{0,\text{loc}}$, we have shown that $\mathcal{M}_{0,\text{loc}}$ is indeed a vector space, as required.

Exercise 7.2 Suppose that $M \in \mathcal{M}_{0,\text{loc}}$ with $[M] \equiv 0$. Show that $M \equiv 0$ in the sense that M is indistinguishable from the 0 process.

Solution 7.2 By Theorem 4.1.9(2), we have that for each $t \geqslant 0$, $\Delta M_t = (\Delta M_t)^2 = 0$, and thus M is continuous, i.e. $M \in \mathcal{M}^c_{0,\text{loc}}$. Now let $(\tau_k)_{k\in\mathbb{N}}$ be a localising sequence for M. For each $k \in \mathbb{N}$, consider the stopping time

$$\sigma_k := \inf\{t \geqslant 0 : |M_t| \geqslant k\}.$$

As $M_0 = 0$ and M is continuous, it follows that $(\sigma_k)_{k\in\mathbb{N}}$ is a sequence of stopping times such that $\sigma_k \uparrow \infty$ P-a.s. By Exercise 7.1(c), we have that $T_k := \tau_k \wedge \sigma_k$ is a localising sequence for M. Note that by construction of σ_k, the martingale M^{T_k} is bounded by k, and in particular M^{T_k} is square-integrable. By Theorem 4.1.9(4), we have $[M^{T_k}] = [M]^{T_k} = 0$, and by Theorem 4.1.9(5), we have that $(M^{T_k})^2 = (M^{T_k})^2 - [M^{T_k}]$ is a martingale. So for each $t \geqslant 0$,

$$E[(M^{T_k}_t)^2] = E[(M^{T_k}_0)^2] = 0,$$

and thus $M^{T_k}_t = 0$ P-a.s. Letting $k \to \infty$ shows that $M_t = 0$ P-a.s. By taking a countable intersection we can see that

$$P[M_t = 0 \text{ for all } t \in \mathbb{Q}_+] = 1.$$

Since M is continuous, this implies that M is indistinguishable from the 0 process, which completes the proof.

Exercise 7.3 Let $M \in \mathcal{H}^2_0$. Show that $b\mathcal{E}$ is dense in $L^2(M)$.

Hint: Equip $\overline{\Omega} = \Omega \times [0,\infty)$ with the predictable σ-algebra $\mathcal{P}$. Let $C := E[M^2_\infty]$ and consider the probability measure $P_M = C^{-1} P \otimes [M]$ on $(\overline{\Omega}, \mathcal{P})$. Let $(\Pi_n)_{n\in\mathbb{N}}$ be an increasing sequence of partitions of $[0,\infty)$ with $\lim_{n\to\infty} |\Pi_n| = 0$. Use the martingale convergence theorem on $(\overline{\Omega}, \mathcal{P}, P_M)$ with respect to the discrete filtration $(\mathcal{P}_n)_{n\in\mathbb{N}}$ defined by

$$\mathcal{P}_n := \sigma(\{A_i \times (t_i, t_{i+1}] : t_i \in \Pi_n, A_i \in \mathcal{F}_{t_i}\}).$$

Solution 7.3 We first note that $L^2(M) = L^2_{P_M}$, since both are equal to the set of (equivalence classes of) predictable processes $\widetilde{H}$ such that

$$\|\widetilde{H}\|^2_{L^2(M)} = E\left[\int_0^\infty \widetilde{H}^2_s d\langle M\rangle_s\right] = C E_M[\widetilde{H}^2] = C\|\widetilde{H}\|^2_{L^2_{P_M}} < \infty.$$

Let $H \in L^2(M)$. We want to approximate H by elements of $b\mathcal{E}$. Since $H\mathbf{1}_{\{|H|\leq n\}} \to H$ in $L^2(M)$ by the dominated convergence theorem, we only need to approximate each $H\mathbf{1}_{\{|H|\leqslant n\}}$. Thus we assume without loss of generality that H is bounded.

Define a P_M-martingale $(H_n)_{n\in\mathbb{N}}$ adapted to $(\mathcal{P}_n)_{n\in\mathbb{N}}$ by $H^n := E_M[H \,|\, \mathcal{P}_n]$. Since $H \in L^2_{P_M} = L^2(M)$, we have that $(H^n)_{n\in\mathbb{N}}$ is an $L^2_{P_M}$-bounded martingale. Let $\mathcal{P}_\infty = \sigma(\bigcup_{n\in\mathbb{N}} \mathcal{P}_n)$ and $H^\infty := E[H \,|\, \mathcal{P}_\infty]$. By the martingale convergence theorem, we have that $H^n \to H^\infty$ P_M-a.s. and in $L^2_{P_M}$.

We claim that $\mathcal{P}_\infty = \mathcal{P}$ and that $H^n \in b\mathcal{E}$ for each $n \in \mathbb{N}$. If this holds, then we can approximate $H = H^\infty$ in $L^2_{P_M} = L^2(M)$ as the limit of $(H^n)_{n\in\mathbb{N}}$, where $H^n \in b\mathcal{E}$ for each $n \in \mathbb{N}$. Thus, the two claims imply the result.

To show that $\mathcal{P}_\infty = \mathcal{P}$, we first note that $\mathcal{P}_n \subseteq \mathcal{P}$ for each $n \in \mathbb{N}$. Indeed, let $\widetilde{H} = \mathbf{1}_{A_i}\mathbf{1}_{(t_i,t_{i+1}]}$ for some $t_i \in \Pi_n$ and $A_i \in \mathcal{F}_{t_i}$. As $\widetilde{H}$ is adapted and left-continuous, it is predictable, i.e., $\mathcal{P}$-measurable, and so $\mathcal{P}_n \subseteq \mathcal{P}$. Taking the union gives $\mathcal{P}_\infty \subseteq \mathcal{P}$.

For the reverse inclusion $\mathcal{P} \subseteq \mathcal{P}_\infty$, we show that any left-continuous adapted process $\widetilde{H}$ is $\mathcal{P}_\infty$-measurable. To this end, define

$$\widetilde{H}^n := \sum_{t_i\in\Pi_n} \mathbf{1}_{(t_i,t_{i+1}]}\widetilde{H}_{t_i},$$

which is $\mathcal{P}_n$-measurable, hence also $\mathcal{P}_\infty$-measurable for each $n \in \mathbb{N}$. For all $t \geqslant 0$ and $n \in \mathbb{N}$, we have that $\widetilde{H}^n_t(\omega) = \widetilde{H}_{t(n)}(\omega)$, where $t(n) := \max\{t_i \in \Pi_n : t_i < t\}$. We have that $t(n)$ is increasing in n, since $(\Pi_n)_{n\in\mathbb{N}}$ is an increasing sequence. Moreover, $t(n) \uparrow t$ since $|\Pi_n| \downarrow 0$. As $\widetilde{H}$ is left-continuous, we conclude that $\widetilde{H}^n_t(\omega) \to \widetilde{H}_t(\omega)$ for all $t \geqslant 0$ and $\omega \in \Omega$. Therefore, as each $\widetilde{H}^n$ is $\mathcal{P}_\infty$-measurable, so is $\widetilde{H}$. We have thus shown that $\mathcal{P}_\infty = \mathcal{P}$, as claimed.

It remains to show that $H^n \in b\mathcal{E}$ for each $n \in \mathbb{N}$. For this we give two proofs.

Proof 1: Note that $H^n = E_M[H \,|\, \mathcal{P}_n]$ is bounded since H is. As H^n is $\mathcal{P}_n$-measurable, the result follows if we show that every bounded $\mathcal{P}_n$-measurable process belongs to $b\mathcal{E}$. To this end, we use the monotone class theorem. Let

$$\mathcal{M} := \{\mathbf{1}_{A_i}\mathbf{1}_{(t_i,t_{i+1}]} : t_i \in \Pi_n, A_i \in \mathcal{F}_{t_i}\}$$

and

$$\mathcal{H} := \left\{ \widetilde{H} = \sum_{t_i\in\Pi_n} Z_i \mathbf{1}_{(t_i,t_{i+1}]} : Z_i \text{ bounded and } \mathcal{F}_{t_i}\text{-measurable} \right\}.$$

It is clear that $\mathcal{M}$ is closed under products, generates $\mathcal{P}_n$ and is contained in $\mathcal{H}$. Moreover, $\mathcal{H}$ is a vector space and contains 1. To see that $\mathcal{H}$ is closed under bounded monotone convergence, let $\mathcal{H} \ni \widetilde{H}^m \uparrow \widetilde{H}$. Then, it must be the case that $Z^m_i = \widetilde{H}^m_{t_{i+1}} \uparrow \widetilde{H}_{t_{i+1}} =: Z_i$, where Z_i is bounded and $\mathcal{F}_{t_i}$-measurable. Moreover, since

$$\widetilde{H}_t = \lim_{m\to\infty} \widetilde{H}^m_t = \lim_{m\to\infty} \widetilde{H}^m_{t(n)} = \widetilde{H}_{t(n)}, \quad t \geq 0,$$

we see that

$$\widetilde{H} = \sum_{t_i \in \Pi_n} Z_i \mathbf{1}_{(t_i, t_{i+1}]} \in \mathcal{H}.$$

Therefore, by the monotone class theorem, $\mathcal{H}$ contains all bounded $\mathcal{P}_n$-measurable processes. Since $\mathcal{H} \subseteq b\mathcal{E}$, we have shown that every bounded $\mathcal{P}_n$-measurable process belongs to $b\mathcal{E}$, as required.

Proof 2: We claim that $H^n_t = \sum_{t_i \in \Pi_n} \mathbf{1}_{\{t \in (t_i, t_{i+1}]\}} Z_i$ for each $n \in \mathbb{N}$, where

$$Z_i := \frac{E[\int_{t_i}^{t_{i+1}} H_u \,\mathrm{d}\langle M\rangle_u \,|\, \mathcal{F}_{t_i}]}{E[\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i} \,|\, \mathcal{F}_{t_i}]}.$$

If this holds, then $H^n \in b\mathcal{E}$, as Z_i is bounded and $\mathcal{F}_{t_i}$-measurable.

Set $K^n_t := \sum_{t_i \in \Pi_n} \mathbf{1}_{\{t \in (t_i, t_{i+1}]\}} Z_i$. To show that $K^n = E_M[H \mid \mathcal{P}_n] = H^n$, first note that K^n is $\mathcal{P}_n$-measurable. We also have that $\{A_i \times (t_i, t_{i+1}] : t_i \in \Pi_n, A_i \in \mathcal{F}_{t_i}\}$ is a π-system generating $\mathcal{P}_n$. Therefore, it suffices to check that for each $t_i \in \Pi_n$ and $A_i \in \mathcal{F}_{t_i}$ we have $E_M[\mathbf{1}_{A_i \times (t_i, t_{i+1}]} K^n] = E_M[\mathbf{1}_{A_i \times (t_i, t_{i+1}]} H]$. So we write

$$\begin{aligned}
E_M[\mathbf{1}_{A_i \times (t_i, t_{i+1}]} H] &= C^{-1} E\left[\mathbf{1}_{A_i} \int_{t_i}^{t_{i+1}} H_u \,\mathrm{d}\langle M\rangle_u\right] \\
&= C^{-1} E\left[\mathbf{1}_{A_i} E\left[\int_{t_i}^{t_{i+1}} H_u \,\mathrm{d}\langle M\rangle_u \,\middle|\, \mathcal{F}_{t_i}\right]\right] \\
&= C^{-1} E\left[\int_0^\infty \mathbf{1}_{A_i} \frac{E[\int_{t_i}^{t_{i+1}} H_u \,\mathrm{d}\langle M\rangle_u \,|\, \mathcal{F}_{t_i}]}{E[\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i} \,|\, \mathcal{F}_{t_i}]} \mathbf{1}_{\{s \in (t_i, t_{i+1}]\}} \,\mathrm{d}\langle M\rangle_s\right] \\
&= E_M[\mathbf{1}_{A_i \times (t_i, t_{i+1}]} K^n],
\end{aligned}$$

as claimed. This completes the proof.

Exercise 7.4 For $M \in \mathcal{M}^c_{0,\mathrm{loc}}$, we denote by $L^2_{\mathrm{loc}}(M)$ the space of all predictable processes for which there exists a sequence of stopping times $(\tau_n)_{n \in \mathbb{N}}$ such that $\tau_n \uparrow \infty$ P-a.s. and $E[\int_0^{\tau_n} H_s^2 \,\mathrm{d}\langle M\rangle_s] < \infty$ for each $n \in \mathbb{N}$.

(a) Let H be predictable. Show that

$$H \in L^2_{\mathrm{loc}}(M) \iff \int_0^t H_s^2 \,\mathrm{d}\langle M\rangle_s < \infty \; P\text{-a.s. for each } t \geqslant 0.$$

(b) Show that for any continuous semimartingale X, any adapted RCLL process H and any sequence of partitions $(\Pi_n)_{n \in \mathbb{N}}$ of $[0, \infty)$ with $\lim_{n \to \infty} |\Pi_n| = 0$, we have

$$\int_0^\cdot H_{s-} \,\mathrm{d}X_s = \lim_{n \to \infty} \sum_{t_i \in \Pi_n} H_{t_i} \left(X_{t_{i+1} \wedge \cdot} - X_{t_i \wedge \cdot}\right) \quad \text{ucp},$$

where ucp stands for *uniformly on compacts in probability.*

(c) Find an adapted process with right-continuous paths which is not locally bounded.

Solution 7.4

(a) For the forward direction, let $H \in L^2_{\mathrm{loc}}(M)$ and let $(\tau_n)_{n\in\mathbb{N}}$ be a corresponding localising sequence. So for each $n \in \mathbb{N}$, we have

$$P\Big[\int_0^{\tau_n} H_s^2 \,\mathrm{d}\langle M\rangle_s < \infty\Big] = 1.$$

Fix any $t \geqslant 0$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned}
P\Big[\int_0^t H_s^2 \, d\langle M\rangle_s = \infty\Big] &= P\Big[\Big\{\int_0^t H_s^2 \,\mathrm{d}\langle M\rangle_s = \infty\Big\} \cap \{\tau_n \leqslant t\}\Big] \\
&\quad + P\Big[\Big\{\int_0^t H_s^2 \,\mathrm{d}\langle M\rangle_s = \infty\Big\} \cap \{\tau_n > t\}\Big] \\
&\leqslant P[\tau_n \leqslant t] + P\Big[\Big\{\int_0^{\tau_n} H_s^2 \,\mathrm{d}\langle M\rangle_s = \infty\Big\} \cap \{\tau_n > t\}\Big] \\
&\leqslant P[\tau_n \leqslant t] + P\Big[\int_0^{\tau_n} H_s^2 \,\mathrm{d}\langle M\rangle_s = \infty\Big] \\
&= \; P[\tau_n \leqslant t].
\end{aligned}$$

Thus, we conclude that

$$P\Big[\int_0^t H_s^2 \,\mathrm{d}\langle M\rangle_s = \infty\Big] \leqslant \lim_{n\to\infty} P[\tau_n \leqslant t] = 0.$$

Conversely, let H be predictable such that

$$\int_0^t H_s^2 \,\mathrm{d}\langle M\rangle_s < \infty \quad P\text{-a.s. for each } t \geqslant 0. \tag{1}$$

Consider the sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ defined by

$$\tau_n := \inf\Big\{t \geq 0 \,\Big|\, \int_0^t H_s^2 \,\mathrm{d}\langle M\rangle_s > n\Big\}.$$

From (1), we obtain $\tau_n \uparrow \infty$ P-a.s. Moreover, by the definition of τ_n and the (left)-continuity of $\int H \,\mathrm{d}\langle M\rangle$, we have for each $n \in \mathbb{N}$ that

$$E\Big[\int_0^{\tau_n} H_s^2 \,\mathrm{d}\langle M\rangle_s\Big] \leqslant n < \infty.$$

This completes the proof.

(b) For each $n \in \mathbb{N}$, set

$$H^n := \sum_{t_i \in \Pi_n} H_{t_i} \, \mathbf{1}_{(t_i, t_{i+1}]}.$$

For all $t \geqslant 0$, we have

$$(H^n \bullet X)_t := \int_0^t H^n_s \, \mathrm{d}X_s = \sum_{t_i \in \Pi_n} H_{t_i} \, (X_{t_{i+1} \wedge t} - X_{t_i \wedge t}).$$

By construction of H^n, we have $H^n \to H_-$ pointwise. Also, since H_- is left-continuous and adapted, so is H^*_-, where $H^*_{t-} := \sup_{0 \leqslant s \leqslant t} |H_{s-}|$. Thus H^*_- is also locally bounded. Now we have $|H^n - H_-| \leqslant 2H^*_-$, and so we can apply Theorem 4.2.23 to get the result.

(c) Let $(\Omega, \mathcal{F}, P)$ be a probability space such that there exists a random variable $Z \sim \mathcal{N}(0,1)$ which is $\mathcal{F}$-measurable. Fix some $u > 0$ and consider the process $X = (X_t)_{t \geqslant 0}$ given by

$$X_t := Z \mathbf{1}_{[u,\infty)}(t).$$

Let $\mathbb{F} = \mathbb{F}^X$ be the filtration generated by the process X. By construction, X is right-continuous and $\mathbb{F}$-adapted. Suppose for contradiction that X is locally bounded. Let $(\tau_n)_{n \in \mathbb{N}}$ be a sequence of stopping times such that $\tau_n \uparrow \infty$ P-a.s. and X^{τ_n} is bounded. Since $X_t \equiv 0$ for $t < u$, we have that $\mathcal{F}_t$ is P-trivial for all $t < u$. So for each $t < u$ and $n \in \mathbb{N}$, $P[\tau_n \leqslant t] \in \{0,1\}$ since $\{\tau_n \leqslant t\} \in \mathcal{F}_t$ as τ_n is a stopping time. Now we can write $\{\tau_n < u\} = \bigcup_{m=1}^{\infty} \{\tau_n \leqslant u - \frac{1}{m}\}$, and so $P[\tau_n < u] = \lim_{m \to \infty} P[\tau_n \leqslant u - \frac{1}{m}] \in \{0,1\}$. But as $\{0,1\} \ni P[\tau_n < u] \to 0$ as $n \to \infty$, there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, $P[\tau_n < u] = 0$. In particular, we have $P[\tau_N < u] = 0$, so that $P[\tau_N \geqslant u] = 1$. Therefore, we have that $X^{\tau_N}_u = X_u = Z$. But this implies that Z is bounded, which is a contradiction as $Z \sim \mathcal{N}(0,1)$. This completes the proof.