Coordinator
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Brownian Motion and Stochastic Calculus

Exercise Sheet 8

Submit by 12:00 on Wednesday, April 16 via the course homepage.

Exercise 8.1 Suppose $M \in \mathcal{M}_{0,\mathrm{loc}}$.

(a) Show that if there exists an integrable random variable Z with $|M_t| \leqslant Z$ for all $t \geqslant 0$, then M is a uniformly integrable martingale.

(b) Suppose in addition that M is continuous. Show that the sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ defined by

$$\tau_n := \inf\{t \geqslant 0 : |M_t| \geqslant n\}$$

forms a localising sequence for M, and each τ_n is a stopping time for the (right-continuous and complete) filtration $\mathbb{F}^M$ generated by M.

Exercise 8.2 Fix a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$. For a filtration $\mathbb{G}$, we let $\mathcal{M}^c_{0,\mathrm{loc}}(\mathbb{G})$ denote the space of continuous local $\mathbb{G}$-martingales null at zero. In all four parts below we assume the processes M and N are independent.

(a) Suppose M and N be martingales with respect to their natural filtrations $\mathbb{F}^M$ and $\mathbb{F}^N$, respectively. Show that MN is a martingale with respect to its natural filtration $\mathbb{F}^{MN}$.

(b) Suppose $M, N \in \mathcal{M}^c_{0,\mathrm{loc}}(\mathbb{F})$. Show that also $MN \in \mathcal{M}^c_{0,\mathrm{loc}}(\mathbb{F})$.

(c) Suppose $M \in \mathcal{M}^c_{0,\mathrm{loc}}(\mathbb{F}^M)$ and $N \in \mathcal{M}^c_{0,\mathrm{loc}}(\mathbb{F}^N)$. Show that $MN \in \mathcal{M}^c_{0,\mathrm{loc}}(\mathbb{F}^{MN})$.

(d) Suppose M and N are continuous $\mathbb{F}$-martingales. Show that MN is also an $\mathbb{F}$-martingale.

Remark: There is an example of a filtration $\mathbb{F}$ and (not continuous) independent bounded $\mathbb{F}$-martingales M and N both null at zero such that MN is not a local $\mathbb{F}$-martingale.

Exercise 8.3 Let W be a Brownian motion with respect to its natural filtration. By using Itô's formula, show that the processes $M^{(1)}, M^{(2)}, M^{(3)}$ given by

$$M^{(1)}_t = e^{t/2}\cos W_t, \quad M^{(2)}_t = tW_t - \int_0^t W_u \,\mathrm{d}u, \quad M^{(3)}_t = W_t^3 - 3tW_t$$

are martingales.