# Brownian Motion and Stochastic Calculus

## Exercise Sheet 8

*Submit by 12:00 on Wednesday, April 16 via the course homepage.*

**Exercise 8.1** Suppose $M \in \mathcal{M}_{0,\mathrm{loc}}$.

(a) Show that if there exists an integrable random variable $Z$ with $|M_t| \leqslant Z$ for all $t \geqslant 0$, then $M$ is a uniformly integrable martingale.

(b) Suppose in addition that $M$ is continuous. Show that the sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ defined by

$$\tau_n := \inf\{t \geqslant 0 : |M_t| \geqslant n\}$$

forms a localising sequence for $M$, and each $\tau_n$ is a stopping time for the (right-continuous and complete) filtration $\mathbb{F}^M$ generated by $M$.

**Solution 8.1**

(a) First note that $M$ is adapted because it is a local martingale, and it is integrable because it is dominated by the integrable random variable $Z$. Now let $(\tau_n)_{n\in\mathbb{N}}$ be a localising sequence for $M$. Fix $0 \leqslant s \leqslant t$ and write

$$E[M_{\tau_n\wedge t} \,|\, \mathcal{F}_s] = M_{\tau_n\wedge s}.$$

Since $M_{\tau_n\wedge s} \to M_s$ and $M_{\tau_n\wedge t} \to M_t$ $P$-a.s., the conditional dominated convergence theorem gives

$$E[M_t \,|\, \mathcal{F}_s] = M_s.$$

Thus $M$ is a martingale. To see explicitly that it is uniformly integrable, we write

$$\lim_{K\to\infty} \sup_{t\geqslant 0} E[|M_t| \mathbf{1}_{\{|M_t|\geqslant K\}}] \leqslant \lim_{K\to\infty} E[Z\mathbf{1}_{\{Z\geqslant K\}}] = 0$$

by the dominated convergence theorem. This completes the proof.

(b) Since $M$ is RCLL, we have $\tau_n \uparrow \infty$ $P$-a.s. It remains to show that for each $n \in \mathbb{N}$, the stopped process $M^{\tau_n}$ is a martingale. By the construction of $\tau_n$ and because $M$ is continuous, we have $|M^{\tau_n}| \leqslant n$ so that $M^{\tau_n}$ is bounded. By Exercise 7.1(b), $M^{\tau_n}$ is a local martingale. So by part (a), we have that $M^{\tau_n}$ is a (uniformly integrable) martingale. Thus $(\tau_n)_{n\in\mathbb{N}}$ is a localising sequence for $M$. Since $M$ is continuous, $\tau_n$ is an $\mathbb{F}^M$-stopping time. This completes the proof.

**Exercise 8.2** Fix a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$. For a filtration $\mathbb{G}$, we let $\mathcal{M}^c_{0,\mathrm{loc}}(\mathbb{G})$ denote the space of continuous local $\mathbb{G}$-martingales null at zero. In all four parts below we assume the processes $M$ and $N$ are independent.

(a) Suppose $M$ and $N$ be martingales with respect to their natural filtrations $\mathbb{F}^M$ and $\mathbb{F}^N$, respectively. Show that $MN$ is a martingale with respect to its natural filtration $\mathbb{F}^{MN}$.

(b) Suppose $M, N \in \mathcal{M}^c_{0,\mathrm{loc}}(\mathbb{F})$. Show that also $MN \in \mathcal{M}^c_{0,\mathrm{loc}}(\mathbb{F})$.

(c) Suppose $M \in \mathcal{M}^c_{0,\mathrm{loc}}(\mathbb{F}^M)$ and $N \in \mathcal{M}^c_{0,\mathrm{loc}}(\mathbb{F}^N)$. Show that $MN \in \mathcal{M}^c_{0,\mathrm{loc}}(\mathbb{F}^{MN})$.

(d) Suppose $M$ and $N$ are continuous $\mathbb{F}$-martingales. Show that $MN$ is also an $\mathbb{F}$-martingale.

*Remark: There is an example of a filtration $\mathbb{F}$ and (not continuous) independent bounded $\mathbb{F}$-martingales $M$ and $N$ both null at zero such that $MN$ is not a local $\mathbb{F}$-martingale.*

**Solution 8.2** We follow many of the arguments in the paper *Some particular problems on martingale theory* by A.S. Cherny, which can be found online at http://alexanderchernyy.com/pmt.pdf.

In what follows, we define for each $n \in \mathbb{N}$ the random times $\tau_n$ and $\sigma_n$ by

$$\tau_n := \inf\{t \geqslant 0 : |M_t| \geqslant n\} \quad \text{and} \quad \sigma_n := \inf\{t \geqslant 0 : |N_t| \geqslant n\}.$$

(a) Clearly $MN$ is $\mathbb{F}^{MN}$-adapted, and it is integrable because

$$E[|M_tN_t|] = E[|M_t|]E[|N_t|] < \infty$$

by independence. It remains to check the martingale property. So fix $0 \leqslant s \leqslant t$ and pick $A \in \mathcal{F}^M_s$ and $B \in \mathcal{F}^N_s$. We use independence, the martingale property of $M$ and $N$ and again independence to compute

$$\begin{aligned} E[M_tN_t\mathbf{1}_{A\cap B}] &= E[M_t\mathbf{1}_A N_t\mathbf{1}_B] \\ &= E[M_t\mathbf{1}_A]E[N_t\mathbf{1}_B] \\ &= E[M_s\mathbf{1}_A]E[N_s\mathbf{1}_B] \\ &= E[M_sN_s\mathbf{1}_{A\cap B}]. \end{aligned}$$

So setting $\mathcal{D} := \{A \cap B : A \in \mathcal{F}^M_s, B \in \mathcal{F}^N_s\}$, we have that

$$\mathcal{D} \subseteq \{C \in \mathcal{F}^M_s \vee \mathcal{F}^N_s : E[M_tN_t\mathbf{1}_C] = E[M_sN_s\mathbf{1}_C]\} =: \mathcal{G}.$$

Note that $\mathcal{D}$ is closed under finite intersections and contains $\Omega$. Also, the dominated convergence theorem shows that $\mathcal{G}$ is closed under increasing limits, and clearly $\mathcal{G}$ is closed under differences (meaning that whenever $C, D \in \mathcal{G}$ with

$C \subseteq D$, we have $D \backslash C \in \mathcal{G}$). It follows by the monotone class theorem for sets (see e.g. [Protter, Theorem 6.2]) that $\sigma(\mathcal{D}) \subseteq \mathcal{G}$. But we have $\sigma(\mathcal{D}) = \mathcal{F}_s^M \vee \mathcal{F}_s^N$, which means that

$$E[M_t N_t \,|\, \mathcal{F}_s^M \vee \mathcal{F}_s^N] = M_s N_s.$$

We have thus shown that $MN$ is an $\mathbb{F}^M \vee \mathbb{F}^N$-martingale. As $\mathbb{F}^{MN} \subseteq \mathbb{F}^M \vee \mathbb{F}^N$, it follows from the tower law that $MN$ is also an $\mathbb{F}^{MN}$-martingale, as required.

(b) By Exercise 8.1(b) (which is applicable because $M$ and $N$ are continuous), $(\tau_n)_{n\in\mathbb{N}}$ and $(\sigma_n)_{n\in\mathbb{N}}$ are localising sequences for $M$ and $N$, respectively, so that for each $n \in \mathbb{N}$, $M^{\tau_n}$ and $N^{\sigma_n}$ are bounded $\mathbb{F}$-martingales. Moreover, since $\tau_n$ and $\sigma_n$ are $\mathbb{F}^M$- and $\mathbb{F}^N$-stopping times, respectively, it can be seen from the independence of $M$ and $N$ that $M^{\tau_n}$ and $N^{\sigma_n}$ are also independent. So $M^{\tau_n}$ and $N^{\sigma_n}$ are independent $\mathbb{F}$-martingales, and thus by the tower law they are also independent martingales with respect to their natural filtrations $\mathbb{F}^{M^{\tau_n}}$ and $\mathbb{F}^{N^{\sigma_n}}$, respectively. So by part (a), $M^{\tau_n}N^{\sigma_n}$ is an $\mathbb{F}^{M^{\tau_n}N^{\sigma_n}}$-martingale. Also, $M^{\tau_n}N^{\sigma_n}$ is continuous because $M$ and $N$ are. We therefore have that $\langle M^{\tau_n}, N^{\sigma_n}\rangle \equiv 0$ when computed in $\mathbb{F}^{M^{\tau_n}N^{\sigma_n}}$. But as $\langle M^{\tau_n}, N^{\sigma_n}\rangle$ can be computed pathwise, we also have $\langle M^{\tau_n}, N^{\sigma_n}\rangle$ when computed in $\mathbb{F}$. We thus have

$$\langle M^{\tau_n}, N^{\sigma_n}\rangle = \langle M, N\rangle^{\tau_n \wedge \sigma_n} \equiv 0, \quad \text{for all } n \in \mathbb{N}.$$

As $\tau_n \wedge \sigma_n \uparrow \infty$ as $n \to \infty$, we have $\langle M, N\rangle \equiv 0$ in $\mathbb{F}$. It therefore follows that $MN \in \mathcal{M}_{0,\text{loc}}^{c}(\mathbb{F})$, as required.

(c) Just as in part (b), we can see that $M^{\tau_n}$ and $N^{\sigma_n}$ are independent $\mathbb{F}^{M^{\tau_n}}$- and $\mathbb{F}^{N^{\sigma_n}}$-martingales, respectively. We then argue the same as part (b), but with $\mathbb{F}$ replaced with $\mathbb{F}^{MN}$.

(d) Assume first that $M_0 = N_0 = 0$. Then we have $M^{\tau_n}, N^{\sigma_n} \in \mathcal{M}_{0,\text{loc}}^{c}(\mathbb{F})$, and since $M^{\tau_n}$ and $N^{\sigma_n}$ are independent (by the same reasoning as in part (b)), we can apply part (b) to get that $M^{\tau_n}N^{\sigma_n} \in \mathcal{M}_{0,\text{loc}}^{c}(\mathbb{F})$. As $M^{\tau_n}N^{\sigma_n}$ is also bounded by the construction of $\tau_n$ and $\sigma_n$, $M^{\tau_n}N^{\sigma_n}$ is a bounded $\mathbb{F}$-martingale. So fixing $0 \leqslant s \leqslant t$, we have

$$E[M_t^{\tau_n} N_t^{\sigma_n} \,|\, \mathcal{F}_s] = M_s^{\tau_n} N_s^{\sigma_n},$$

or equivalently,

$$E[M_t^{\tau_n} N_t^{\sigma_n} \mathbf{1}_A] = E[M_s^{\tau_n} N_s^{\sigma_n} \mathbf{1}_A], \quad \text{for all } A \in \mathcal{F}_s. \tag{1}$$

Now note that as $n \to \infty$, $M_t^{\tau_n} \to M_t$ $P$-a.s. By the stopping theorem, we have $M_t^{\tau_n} = E[M_t \,|\, \mathcal{F}_{\tau_n \wedge t}]$, and thus the family $(M_t^{\tau_n})_{n\in\mathbb{N}}$ is uniformly integrable. Therefore we also have that $M_t^{\tau_n} \to M_t$ in $L^1(\Omega)$. Similarly we have $N_t^{\tau_n} \to N_t$ in $L^1(\Omega)$. We now write, using independence of $M$ and $N$,

$$\begin{aligned} E[|M_t^{\tau_n} N_t^{\sigma_n} - M_t N_t|] &\leqslant E[|M_t^{\tau_n}||N_t^{\sigma_n} - N_t|] + E[|M_t^{\tau_n} - M_t||N_t|] \\ &= E[|M_t^{\tau_n}|]\|N_t^{\sigma_n} - N_t\|_{L^1(\Omega)} + \|M_t^{\tau_n} - M_t\|_{L^1(\Omega)} E[|N_t|] \\ &\leqslant C\|N_t^{\sigma_n} - N_t\|_{L^1(\Omega)} + \|M_t^{\tau_n} - M_t\|_{L^1(\Omega)} E[|N_t|], \end{aligned}$$

where in the last step we set $C := \sup_{n\in\mathbb{N}} E[|M_t^{\tau_n}|]$, which we know is finite as $(M_t^{\tau_n})_{n\in\mathbb{N}}$ is uniformly integrable. We have thus shown that $M_t^{\tau_n} N_t^{\sigma_n} \to M_t N_t$ in $L^1(\Omega)$. Similarly we also have $M_s^{\tau_n} N_s^{\sigma_n} \to M_s N_s$ in $L^1(\Omega)$. So for fixed $A \in \mathcal{F}_s$, we also have $M_t^{\tau_n} N_t^{\sigma_n} \mathbf{1}_A \to M_t N_t \mathbf{1}_A$ and $M_s^{\tau_n} N_s^{\sigma_n} \mathbf{1}_A \to M_s N_s \mathbf{1}_A$ in $L^1(\Omega)$. Thus we may take the limit as $n \to \infty$ in (1) to get

$$E[M_t N_t \mathbf{1}_A] = E[M_s N_s \mathbf{1}_A], \quad \text{for all } A \in \mathcal{F}_s,$$

so that $E[M_t N_t \,|\, \mathcal{F}_s] = M_s N_s$. As $0 \leqslant s \leqslant t$ were chosen arbitrarily, we have thus shown that $MN$ is an $\mathbb{F}$-martingale.

Finally, for the general case when $M_0$ and $N_0$ are not necessarily 0, we define the $\mathbb{F}$-martingales $\widetilde{M}$ and $\widetilde{N}$ by $\widetilde{M}_t := M_t - M_0$ and $\widetilde{N}_t := N_t - N_0$. Then we write

$$M_t N_t = M_0 N_0 + M_0 \widetilde{N}_t + \widetilde{M}_t N_0 + \widetilde{M}_t \widetilde{N}_t. \tag{2}$$

From above we know that $\widetilde{M}\widetilde{N}$ is an $\mathbb{F}$-martingale, and so from (2) it can be easily seen that $MN$ is also an $\mathbb{F}$-martingale, completing the proof.

**Exercise 8.3** Let $W$ be a Brownian motion with respect to its natural filtration. By using Itô's formula, show that the processes $M^{(1)}, M^{(2)}, M^{(3)}$ given by

$$M_t^{(1)} = e^{t/2} \cos W_t, \quad M_t^{(2)} = tW_t - \int_0^t W_u \,\mathrm{d}u, \quad M_t^{(3)} = W_t^3 - 3tW_t$$

are martingales.

**Solution 8.3** We can express $M^{(1)}, M^{(2)}, M^{(3)}$ in the form

$$M_t^{(1)} = f^{(1)}(t, W_t), \quad M_t^{(2)} = f^{(2)}\left(t, W_t, \int_0^t W_u \,\mathrm{d}u\right), \quad M_t^{(3)} = f^{(3)}(t, W_t),$$

where

$$f^{(1)}(t,w) = e^{t/2} \cos w, \quad f^{(2)}(t,w,x) = tw - x, \quad f^{(3)}(t,w) = w^3 - 3tw$$

are $C^2$ functions. We note that the processes $I$ and $X$ defined by $I_t = t$ and $X_t = \int_0^t W_u \,\mathrm{d}u$, respectively, are continuous and have finite variation, while $\langle W \rangle_t = t$. Therefore we have by Itô's formula that

$$\begin{aligned} M_t^{(1)} &= M_0^{(1)} + \int_0^t \frac{\partial f^{(1)}}{\partial t}(s, W_s) \,\mathrm{d}s + \int_0^t \frac{\partial f^{(1)}}{\partial w}(s, W_s) \,\mathrm{d}W_s + \frac{1}{2} \int_0^t \frac{\partial^2 f^{(1)}}{\partial w^2}(s, W_s) \,\mathrm{d}s \\ &= 1 + \frac{1}{2} \int_0^t e^{s/2} \cos W_s \,\mathrm{d}s - \int_0^t e^{s/2} \sin W_s \,\mathrm{d}W_s - \frac{1}{2} \int_0^t e^{s/2} \cos W_s \,\mathrm{d}s \\ &= 1 - \int_0^t e^{s/2} \sin W_s \,\mathrm{d}W_s, \end{aligned}$$

$$
\begin{aligned}
M_t^{(2)} &= M_0^{(2)} + \int_0^t \frac{\partial f^{(2)}}{\partial t}(s, W_s, X_s)\,\mathrm{d}s + \int_0^t \frac{\partial f^{(2)}}{\partial w}(s, W_s, X_s)\,\mathrm{d}W_s \\
&\quad + \int_0^t \frac{\partial f^{(2)}}{\partial x}(s, W_s, X_s)\,\mathrm{d}X_s + \frac{1}{2}\int_0^t \frac{\partial^2 f^{(2)}}{\partial w^2}(s, W_s, X_s)\,\mathrm{d}s \\
&= \int_0^t W_s\,\mathrm{d}s + \int_0^t s\,\mathrm{d}W_s - X_t \\
&= \int_0^t s\,\mathrm{d}W_s, \\
M_t^{(3)} &= M_0^{(3)} + \int_0^t \frac{\partial f^{(3)}}{\partial t}(s, W_s)\,\mathrm{d}s + \int_0^t \frac{\partial f^{(3)}}{\partial w}(s, W_s)\,\mathrm{d}W_s + \frac{1}{2}\int_0^t \frac{\partial^2 f^{(3)}}{\partial w^2}(s, W_s)\,\mathrm{d}s \\
&= -3\int_0^t W_s\,\mathrm{d}s + \int_0^t (3W_s^2 - 3s)\,\mathrm{d}W_s + 3\int_0^t W_s\,\mathrm{d}s \\
&= \int_0^t (3W_s^2 - 3s)\,\mathrm{d}W_s.
\end{aligned}
$$

Since the integrands are continuous, hence locally bounded, we immediately get that $M^{(1)}, M^{(2)}, M^{(3)}$ are local martingales. To show that they are martingales, remember that $W_t^* \overset{(\mathrm{d})}{=} |W_t|$. Since all moments of the Gaussian distribution are finite, the same is true of $W_t^*$. Therefore,

$$
\begin{aligned}
E\left[\int_0^T e^s (\sin W_s)^2\,\mathrm{d}\langle W\rangle_s\right] &= E\left[\int_0^T e^s (\sin W_s)^2\,\mathrm{d}s\right] \leqslant \int_0^T e^s\,\mathrm{d}s = e^T - 1 < \infty, \\
E\left[\int_0^T s^2\,\mathrm{d}\langle W\rangle_s\right] &= \int_0^T s^2\,\mathrm{d}s = T^3/3 < \infty, \\
E\left[\int_0^T (3W_s^2 - 3s)^2\,\mathrm{d}\langle W\rangle_s\right] &= E\left[\int_0^T (3W_s^2 - 3s)^2\,\mathrm{d}s\right] \leqslant T E[(3(W_T^*)^2 + 3T)^2] < \infty,
\end{aligned}
$$

which shows that $(M^{(1)})^T, (M^{(2)})^T, (M^{(3)})^T \in \mathcal{H}^{2,c}$ for any $T > 0$. In particular, $M^{(1)}, M^{(2)}$ and $M^{(3)}$ are martingales, as required.