Coordinator
Benjamin Kotlov

Brownian Motion and Stochastic Calculus

Exercise Sheet 9

Submit by 12:00 on Wednesday, April 30 via the course homepage.

Exercise 9.1 Let W be a Brownian motion in $\mathbb{R}$ and $\mu \neq 0$ a constant. Define the processes $X^1 = W$ and $X_t^2 = W_t + \mu t$, $t \geqslant 0$, and let P and Q denote the laws on $C[0, \infty)$ of X^1 and X^2, respectively. Prove that $P \perp Q$, meaning that P and Q are mutually singular. Conclude that $Q \not\ll P$ and $Q \not\approx P$. Show however that $Q \overset{\text{loc}}{\approx} P$, and write out explicitly $\frac{\mathrm{d}Q|_{\mathcal{F}_t}}{\mathrm{d}P|_{\mathcal{F}_t}}$.

Exercise 9.2 Let $B = (B^1, \dots, B^n)$ be a Brownian motion in $\mathbb{R}^n$ starting at $y \neq 0$, where $n \geqslant 2$, and set $X := |B|$. The process X is called the *Bessel process of order* n.

(a) Show that there exists some Brownian motion W (not necessarily with respect to the same filtration as for B) such that

$$\mathrm{d}X_t = \mathrm{d}W_t + \frac{n-1}{2X_t}\,\mathrm{d}t.$$

Hint: You may use that $P[B_t \neq 0 \text{ for all } t \geqslant 0] = 1$.

Remark: By using mollifiers, one may show the same result when $y = 0$.

(b) Define the process $\overline{X} = |B|^2$. Show that for the same Brownian motion W as in part (a), we have

$$\mathrm{d}\overline{X}_t = 2\sqrt{\overline{X}_t}\,\mathrm{d}W_t + n\,\mathrm{d}t.$$

Exercise 9.3 Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geqslant 0}, P)$ be a filtered probability space satisfying the usual conditions.

(a) Let $W, \widetilde{W}$ be two $(P, \mathbb{F})$-Brownian motions. Show that $\mathrm{d}\langle W, \widetilde{W}\rangle_t = \rho_t\,\mathrm{d}t$ for some predictable process ρ taking values in $[-1, 1]$.

Hint: Use the Kunita–Watanabe decomposition.

(b) The filtration $\mathbb{F}$ is called *P-continuous* if all local $(P, \mathbb{F})$-martingales are continuous. Show that $\mathbb{F}$ is P-continuous if and only if $\mathbb{F}$ is Q-continuous for all $Q \approx P$.

(c) Assume $\mathbb{F}$ is P-continuous and $Q \approx P$. Show that each local $(Q, \mathbb{F})$-martingale $S = (S_t)_{t \geqslant 0}$ is of the form

$$S_t = S_0 + M_t + \int_0^t \alpha_s \,\mathrm{d}\langle M \rangle_s \quad \text{with } \alpha \in L^2_{\mathrm{loc}}(M) \tag{1}$$

for some $M \in \mathcal{M}^c_{0,\mathrm{loc}}(P)$.

Hint: Use Girsanov's theorem to find a semimartingale decomposition for S under P. Then use the Kunita–Watanabe decomposition under P to describe its finite variation part.

Remark: If S has the form (2), one says that it satisfies the structure condition. This is a useful concept in mathematical finance.

Exercise 9.4 Let $B = (B^1, B^2, B^3)$ be a Brownian motion in $\mathbb{R}^3$ and fix a standard normal random variable $Z = (Z^1, Z^2, Z^3)$ independent of B. Define the process $M = (M_t)_{t \geqslant 0}$ by

$$M_t = \frac{1}{|Z + B_t|}.$$

(a) Show that $P[B_t \neq -Z \text{ for all } t \geqslant 0] = 1$ so that M is a.s. well defined.

Hint: You may use that $P[B_t \neq x \text{ for all } t \geqslant 0] = 1$ for any $x \in \mathbb{R}^3 \setminus \{0\}$.

(b) Show that $|Z + B_t|^2 \sim \mathrm{Gamma}(\frac{3}{2}, \frac{1}{2(t+1)})$ for each $t > 0$, i.e., its density is given by

$$f_t(y) = \frac{(2(t+1))^{-3/2} y^{1/2}}{\Gamma(3/2)} \exp\left(-\frac{y}{2(t+1)}\right), \quad y \geqslant 0.$$

Hint: Recall that when $Y_1, \dots, Y_n \sim$ Gamma(α, β) are independent, we have $Y_1 + \dots + Y_n \sim$ Gamma$(n\alpha, \beta)$.

(c) Show that M is a continuous local martingale. Moreover, show that M is bounded in L^2, i.e., $\sup_{t \geqslant 0} E[|M_t|^2] < \infty$.

(d) Show that M is a strict local martingale, i.e., M is not a martingale.

Remark: This is a standard example of a local martingale which is not a (true) martingale. It also shows that even boundedness in L^2 (which implies uniform integrability) does not guarantee the martingale property.

Exercise 9.5 Consider a probability space $(\Omega, \mathcal{F}, P)$ supporting a Brownian motion $W = (W_t)_{t \geqslant 0}$. Denote by $\mathbb{F} = (\mathcal{F}_t)_{t \geqslant 0}$ the P-augmentation of the raw filtration generated by W. Moreover, fix $T > 0$, $a < b$, and let $F := \mathbf{1}_{\{a \leqslant W_T \leqslant b\}}$. The aim of this exercise is to find explicitly the integrand $H \in L^2_{\mathrm{loc}}(W)$ in the Itô representation

$$F = E[F] + \int_0^T H_s \,\mathrm{d}W_s. \tag{$*$}$$

(a) Define the martingale $M = (M_t)_{0 \leqslant t \leqslant T}$ by $M_t := E[F \mid \mathcal{F}_t]$. Show that there exists a C^2-function $g : \mathbb{R} \times [0,T) \to \mathbb{R}$ such that

$$M_t = g(W_t, t), \quad 0 \leqslant t < T,$$

Compute g explicitly in terms of the distribution function Φ of the standard normal distribution.

(b) Let $(t_n)_{n \in \mathbb{N}}$ be a sequence of nonnegative reals with $t_n \uparrow T$. Use Itô's formula to find for each $n \in \mathbb{N}$ a predictable process H^n such that

$$M^{t_n} - M_0 = H^n \bullet W.$$

(c) Find the process $H \in L^2_{\mathrm{loc}}(W)$ on $[0,T]$ satisfying $(*)$.