Brownian Motion and Stochastic Calculus

Exercise Sheet 9

Submit by 12:00 on Wednesday, April 30 via the course homepage.

Exercise 9.1 Let W be a Brownian motion in $\mathbb{R}$ and $\mu \neq 0$ a constant. Define the processes $X^1 = W$ and $X_t^2 = W_t + \mu t$, $t \geqslant 0$, and let P and Q denote the laws on $C[0,\infty)$ of X^1 and X^2, respectively. Prove that $P \perp Q$, meaning that P and Q are mutually singular. Conclude that $Q \not\ll P$ and $Q \not\approx P$. Show however that $Q \overset{\text{loc}}{\approx} P$, and write out explicitly $\frac{\mathrm{d}Q|_{\mathcal{F}_t}}{\mathrm{d}P|_{\mathcal{F}_t}}$.

Solution 9.1 Define the set $A \subseteq C[0,\infty)$ by

$$A := \left\{ x \in C[0,\infty) : \lim_{t\to\infty} \frac{x(t)}{t} = 0 \right\}.$$

We know from the lecture notes that almost surely,

$$\lim_{t\to\infty} \frac{W_t}{t} = 0.$$

and therefore almost surely,

$$\lim_{t\to\infty} \frac{X_t^1}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{X_t^2}{t} = \mu \neq 0.$$

It follows that $P[A] = 1$ and $Q[A^c] = 1$, so that $P \perp Q$. It follows that $Q \not\ll P$, and thus also $Q \not\approx P$, since $P[A^c] = 0$ but $Q[A^c] = 1 \neq 0$. However, we do have that $Q \overset{\text{loc}}{\approx} P$. Indeed, fix $t \geqslant 0$ and define the process $b \equiv -\mu \in L^2_{\text{loc}}(W)$. We set $Z = \mathcal{E}(\int b_s \,\mathrm{d}W_s) = \mathcal{E}(-\mu W) = (\exp(-\mu W_s - \frac{1}{2}\mu^2 s))_{0\leqslant s\leqslant t}$ and define the probability measure Q^0 on $\mathcal{F}_t$ by

$$\frac{\mathrm{d}Q^0|_{\mathcal{F}_t}}{\mathrm{d}P|_{\mathcal{F}_t}} = Z_t. \tag{1}$$

As $Z > 0$ P-a.s., we have $Q^0 \approx P$ on $\mathcal{F}_t$. By Theorem 4.10, W is under Q^0 on $[0,t]$ a Brownian motion with drift $-\mu$, i.e. $W_s = \widetilde{W}_s - \mu s$ for $0 \leqslant s \leqslant t$ for some Q^0-Brownian motion $\widetilde{W}$ on $[0,t]$. But rearranging gives $\widetilde{W}_s = W_s + \mu s = X_s^2$. So on $[0,t]$, X^2 has under Q and under A^0 the same distribution, and so $P|_{\mathcal{F}_t} \approx Q^0|_{\mathcal{F}_t} = Q|_{\mathcal{F}_t}$. As t was arbitrary, this shows that $Q \approx P$ with

$$\frac{\mathrm{d}Q|_{\mathcal{F}_t}}{\mathrm{d}P|_{\mathcal{F}_t}} = Z_t.$$

Note, however, that $Q^0 = Q^{0,t}$ as defined in (1) is an entire family of measures depending on t; this family is consistent, but its Kolmogorov extension to $\mathcal{F}_\infty$ is not equivalent to P on $\mathcal{F}_\infty$. Indeed, $\lim_{t\to\infty} \frac{1}{t} \log Z_t = -\frac{1}{2}\mu^2 < 0$ implies that $\lim_{t\to\infty} Z_t = 0$.

Exercise 9.2 Let $B = (B^1, \dots, B^n)$ be a Brownian motion in $\mathbb{R}^n$ starting at $y \neq 0$, where $n \geqslant 2$, and set $X := |B|$. The process X is called the *Bessel process of order* n.

(a) Show that there exists some Brownian motion W (not necessarily with respect to the same filtration as for B) such that

$$\mathrm{d}X_t = \mathrm{d}W_t + \frac{n-1}{2X_t}\,\mathrm{d}t.$$

Hint: You may use that $P[B_t \neq 0$ for all $t \geqslant 0] = 1$.

Remark: By using mollifiers, one may show the same result when $y = 0$.

(b) Define the process $\overline{X} = |B|^2$. Show that for the same Brownian motion W as in part (a), we have

$$\mathrm{d}\overline{X}_t = 2\sqrt{\overline{X}_t}\,\mathrm{d}W_t + n\,\mathrm{d}t.$$

Solution 9.2

(a) Consider the C^2-function $f : \mathbb{R}^n \backslash \{0\} \to (0, \infty)$ given by $f(x) := |x|$. By the hint, we can write $X = f(B)$ P-a.s. Applying Itô's formula then gives

$$dX_t = |y| + \sum_{i=1}^n \int_0^t \frac{B_s^i}{|B_s|}\,\mathrm{d}B_s^i + \frac{n-1}{2}\int_0^t \frac{1}{|B_s|}\,\mathrm{d}s.$$

Indeed, this can be seen by writing $f(x_1, \dots, x_n) = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$ and computing

$$\frac{\partial f}{\partial x_i}(x_1, \dots, x_n) = \frac{1}{2}\left(\sum_{i=1}^n x_i^2\right)^{-\frac{1}{2}} 2x_i = \frac{x_i}{|x|}$$

and

$$\frac{\partial^2 f}{\partial (x_i)^2}(x_1, \dots, x_n) = \frac{|x| - x_i \frac{x_i}{|x|}}{|x|^2} = \frac{|x|^2 - |x_i|^2}{|x|^3} = \frac{1}{|x|} - \frac{|x_i|^2}{|x|^3},$$

so that

$$\sum_{i=1}^n \frac{\partial^2 f}{\partial (x_i)^2}(x_1, \dots, x_n) = \frac{n}{|x|} - \frac{\sum_{i=1}^n |x_i|^2}{|x|^3} = \frac{n-1}{|x|}.$$

Setting $W_t := \sum_{i=1}^n \int_0^t \frac{B_s^i}{|B_s|}\,\mathrm{d}B_s^i$, it remains to show that $W = (W_t)_{t\geqslant 0}$ is a Brownian motion. By construction, W is a continuous local martingale null at zero. Moreover, as $\langle B^i, B^j \rangle_t = \delta_{i,j} t$, we get that

$$\langle W \rangle_t = \sum_{i=1}^d \int_0^t \frac{|B_s^i|^2}{|B_s|^2} \,\mathrm{d}s = t.$$

We may therefore conclude by Lévy's characterisation of Brownian motion that W is a Brownian motion, as required.

Alternatively, we can argue as follows. We know from the lecture notes that $\mathrm{d}\overline{X}_t = 2\sqrt{\overline{X}_t}\,\mathrm{d}W_t + n\,\mathrm{d}t$. So Itô's formula with $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ gives for $X = f(\overline{X})$ that

$$\begin{aligned}
\mathrm{d}X_t &= \frac{1}{2}(\overline{X}_t)^{-\frac{1}{2}}\,\mathrm{d}\overline{X}_t - \frac{1}{8}(\overline{X}_t)^{-\frac{3}{2}}\,\mathrm{d}\langle \overline{X} \rangle_t \\
&= \frac{1}{2X_t}(2X_t\,\mathrm{d}W_t + n\,\mathrm{d}t) - \frac{1}{8}X_t^{-3}4X_t^2\,\mathrm{d}t \\
&= \mathrm{d}W_t + \frac{n-1}{2X_t}\,\mathrm{d}t,
\end{aligned}$$

as required.

(b) Applying Itô's formula to $\overline{X} = g(X)$, where $g(x) = x^2$, gives us

$$\mathrm{d}\overline{X}_t = 2X_t\,\mathrm{d}X_t + \mathrm{d}\langle X \rangle_t.$$

From part (a), we know that $\mathrm{d}X_t = \mathrm{d}W_t + \frac{n-1}{2X_t}\,\mathrm{d}t$, and therefore $\mathrm{d}\langle X \rangle_t = \mathrm{d}t$. We thus obtain

$$\mathrm{d}\overline{X}_t = 2X_t\,\mathrm{d}W_t + (n-1)\,\mathrm{d}t + \mathrm{d}t = 2X_t\,\mathrm{d}W_t + n\,\mathrm{d}t.$$

Writing $X_t = \sqrt{\overline{X}_t}$ then gives the result.

Exercise 9.3 Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}, P)$ be a filtered probability space satisfying the usual conditions.

(a) Let $W, \widetilde{W}$ be two $(P, \mathbb{F})$-Brownian motions. Show that $\mathrm{d}\langle W, \widetilde{W} \rangle_t = \rho_t\,\mathrm{d}t$ for some predictable process ρ taking values in $[-1, 1]$.

Hint: Use the Kunita–Watanabe decomposition.

(b) The filtration $\mathbb{F}$ is called *P-continuous* if all local $(P, \mathbb{F})$-martingales are continuous. Show that $\mathbb{F}$ is P-continuous if and only if $\mathbb{F}$ is Q-continuous for all $Q \approx P$.

(c) Assume $\mathbb{F}$ is P-continuous and $Q \approx P$. Show that each local $(Q, \mathbb{F})$-martingale $S = (S_t)_{t\geqslant 0}$ is of the form

$$S_t = S_0 + M_t + \int_0^t \alpha_s\,\mathrm{d}\langle M \rangle_s \quad \text{with } \alpha \in L^2_{\mathrm{loc}}(M) \tag{2}$$

for some $M \in \mathcal{M}^c_{0,\mathrm{loc}}(P)$.

Hint: Use Girsanov's theorem to find a semimartingale decomposition for S under P. Then use the Kunita–Watanabe decomposition under P to describe its finite variation part.

Remark: If S has the form (2), one says that it satisfies the structure condition. This is a useful concept in mathematical finance.

Solution 9.3

(a) Using the Kunita–Watanabe decomposition, we can write $W = \rho \bullet \widetilde{W} + N$ for some predictable integrand $\rho \in L^2_{\mathrm{loc}}(\widetilde{W})$ and some local martingale $N \in \mathcal{M}^c_{0,\mathrm{loc}}$ strongly orthogonal to $\widetilde{W}$. Then by orthogonality and associativity of the stochastic integral, we have

$$\langle W, \widetilde{W}\rangle_t = \langle \rho \bullet \widetilde{W} + N, \widetilde{W}\rangle_t = \int_0^t \rho_s \,\mathrm{d}\langle \widetilde{W}\rangle_s + \langle N, \widetilde{W}\rangle_t = \int_0^t \rho_s \,\mathrm{d}s, \quad t \geqslant 0.$$

It remains to show that ρ takes values in $[-1, 1]$. To this end, we first write for each $t \geqslant 0$

$$\begin{aligned}\langle \rho \bullet \widetilde{W} + N\rangle_t &= \int_0^t \rho_s^2 \,\mathrm{d}s + 2\int_0^t \rho_s \,\mathrm{d}\langle \widetilde{W}, N\rangle_s + \langle N\rangle_t \\ &= \int_0^t \rho_s^2 \,\mathrm{d}s + \langle N\rangle_t,\end{aligned}$$

where in the last line we use that $\widetilde{W}$ and N are strongly orthogonal. We therefore have

$$\int_0^t \mathrm{d}s = t = \langle W\rangle_t = \langle \rho \bullet \widetilde{W} + N\rangle_t = \int_0^t \rho_s^2 \,\mathrm{d}s + \langle N\rangle_t,$$

and hence

$$\int_0^t (1 - \rho_s^2) \,\mathrm{d}s = \langle N\rangle_t.$$

In particular, the process $t \mapsto \int_0^t (1-\rho_s^2)\,\mathrm{d}s$ is increasing, implying that $\rho^2 \leqslant 1$ $\mathrm{d}t \otimes P$-a.e., as required.

(b) We only need to show the implication "$\Rightarrow$", as "$\Leftarrow$" is trivial by taking $Q = P$. So fix $Q \approx P$ and let $Z^Q = (Z^Q_t)_{t\geqslant 0}$ be the density process of Q with respect to P. Since Z^Q is a $(P, \mathbb{F})$-martingale, Z^Q is continuous. Since $Q \approx P$, we have that $Z^Q_t > 0$ P-a.s. and Q-a.s. for each $t \geqslant 0$ a.s. Therefore, $1/Z^Q$ is also continuous.

Now let X be a local $(Q, \mathbb{F})$-martingale. Then $Z^Q X$ is a local $(P, \mathbb{F})$-martingale and thus continuous P-a.s. Therefore, $X = \frac{1}{Z^Q}(Z^Q X)$ is continuous P-a.s. As $Q \approx P$, we have that X is also continuous Q-a.s. Since X is an arbitrary local $(Q, \mathbb{F})$-martingale, we have shown that $\mathbb{F}$ is Q-continuous, completing the proof.

(c) Let $Z^{P;Q}$ be the density process of P with respect to Q. Note that $Z_0^{P;Q} = 1$, and moreover $Z^{P;Q}$ is strictly positive and continuous by part (b). Therefore, we can write $Z^{P;Q} = \mathcal{E}(L)$ for $L \in \mathcal{M}^c_{0,\mathrm{loc}}(Q)$ defined by $L = \frac{1}{Z^{P;Q}} \bullet Z_P$.

Since S is a continuous local Q-martingale, we obtain by Girsanov's theorem that the process M given by

$$M := S - S_0 - \langle L, S - S_0 \rangle$$

is a continuous local P-martingale. Rewriting, we get the P-semimartingale decomposition

$$S = S_0 + M + \langle L, S - S_0 \rangle.$$

It thus only remains to show that $\langle L, S - S_0 \rangle = \int \alpha \, \mathrm{d}\langle M \rangle$ for some $\alpha \in L^2_{\mathrm{loc}}(M)$. Since $L \in \mathcal{M}^c_{0,\mathrm{loc}}(Q)$, Girsanov's theorem gives $\widetilde{L} := L - \langle L \rangle \in \mathcal{M}^c_{0,\mathrm{loc}}(P)$. Applying the Kunita–Watanabe decomposition to $\widetilde{L}$ with respect to M, we obtain that $\widetilde{L} = \int \alpha \, \mathrm{d}M + N$ for some $\alpha \in L^2_{\mathrm{loc}}(M)$ and some $N \in \mathcal{M}^c_{0,\mathrm{loc}}(P)$ with $N \perp M$. Since $M - (S - S_0)$ and $\widetilde{L} - L$ are continuous finite variation processes, their quadratic covariation is 0. Therefore,

$$\langle L, S - S_0 \rangle = \langle \widetilde{L}, M \rangle = \left\langle \int \alpha \, \mathrm{d}M + N, M \right\rangle = \int \alpha \, \mathrm{d}\langle M \rangle,$$

which completes the proof.

Exercise 9.4 Let $B = (B^1, B^2, B^3)$ be a Brownian motion in $\mathbb{R}^3$ and fix a standard normal random variable $Z = (Z^1, Z^2, Z^3)$ independent of B. Define the process $M = (M_t)_{t \geqslant 0}$ by

$$M_t = \frac{1}{|Z + B_t|}.$$

(a) Show that $P[B_t \neq -Z \text{ for all } t \geqslant 0] = 1$ so that M is a.s. well defined.

Hint: You may use that $P[B_t \neq x$ for all $t \geqslant 0] = 1$ for any $x \in \mathbb{R}^3 \setminus \{0\}$.

(b) Show that $|Z + B_t|^2 \sim \mathrm{Gamma}(\frac{3}{2}, \frac{1}{2(t+1)})$ for each $t > 0$, i.e., its density is given by

$$f_t(y) = \frac{(2(t+1))^{-3/2} y^{1/2}}{\Gamma(3/2)} \exp\left(-\frac{y}{2(t+1)}\right), \quad y \geqslant 0.$$

Hint: Recall that when $Y_1, \dots, Y_n \sim$ Gamma(α, β) are independent, we have $Y_1 + \dots + Y_n \sim$ Gamma$(n\alpha, \beta)$.

(c) Show that M is a continuous local martingale. Moreover, show that M is bounded in L^2, i.e., $\sup_{t \geqslant 0} E[|M_t|^2] < \infty$.

(d) Show that M is a strict local martingale, i.e., M is not a martingale.

Remark: This is a standard example of a local martingale which is not a (true) martingale. It also shows that even boundedness in L^2 (which implies uniform integrability) does not guarantee the martingale property.

Solution 9.4

(a) By independence of B and Z, we have that

$$\begin{aligned} P[B_t \neq -Z \text{ for all } t \geqslant 0] &= E\Big[P[B_t \neq -x \text{ for all } t \geqslant 0]|_{x=Z}\Big] \\ &\geqslant E[\mathbf{1}_{\{Z \neq 0\}}] \\ &= P[Z \neq 0] \\ &= 1, \end{aligned}$$

as required.

(b) We first find the density function $\widetilde{f}_t$ of $|Z^1 + B^1_t|^2$. As $Z^1 + B^1_t \sim \mathcal{N}(0, t+1)$ by independence, we have for each $y \geqslant 0$ that

$$\begin{aligned} P[|Z^1 + B^1_t|^2 \leqslant y] &= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi(t+1)}} e^{-\frac{z^2}{2(t+1)}} \,\mathrm{d}z. \\ &= 2\int_0^{\sqrt{y}} \frac{1}{\sqrt{2\pi(t+1)}} e^{-\frac{z^2}{2(t+1)}} \,\mathrm{d}z. \end{aligned}$$

Changing variables to $u = z^2$, we find that

$$P[|Z^1 + B^1_t|^2 \leqslant y] = 2\int_0^{y} \frac{1}{\sqrt{2\pi(t+1)}} e^{-\frac{u}{2(t+1)}} \frac{1}{2\sqrt{u}} \,\mathrm{d}u.$$

Differentiating in y, we get that

$$\widetilde{f}_t(y) = \frac{y^{-1/2}}{(2(t+1))^{1/2}\sqrt{\pi}} e^{-\frac{y}{2(t+1)}} = \frac{y^{-1/2}}{(2(t+1))^{1/2}\Gamma(1/2)} e^{-\frac{y}{2(t+1)}}.$$

Therefore, $|Z^1 + B^1_t|^2 \sim \text{Gamma}(\frac{1}{2}, \frac{1}{2(t+1)})$, and similarly for $|Z^2 + B^2_t|^2$ and $|Z^3 + B^3_t|^2$.

Now since $Z^1 + B^1, Z^2 + B^2, Z^3 + B^3$ are independent and $\sim \text{Gamma}(\frac{1}{2}, \frac{1}{2(t+1)})$, we have by the hint that

$$|Z + B_t|^2 = |Z^1 + B^1_t|^2 + |Z^2 + B^2_t|^2 + |Z^3 + B^3_t|^2 \sim \text{Gamma}\left(\frac{3}{2}, \frac{1}{2(t+1)}\right),$$

as required.

(c) By part (a), the process $Z + B = (Z + B_t)_{t\geqslant 0}$ takes values in the open set $D := \mathbb{R}^3 \setminus \{0\}$ P-a.s. So we can apply Itô's formula to $M_t = f(B_t)$ with $f : D \to (0,\infty)$ given by $f(y) := \frac{1}{|y|}$.

For $i = 1, 2, 3$, we have

$$\frac{\partial f}{\partial y^i}(y) = -\frac{y^i}{|y|^3}, \quad \frac{\partial^2 f}{(\partial y^i)^2}(y) = \frac{-|y|^2 + 3(y^i)^2}{|y|^5}.$$

It follows that $\Delta f = \frac{\partial^2 f}{(\partial y^1)^2} + \frac{\partial^2 f}{(\partial y^2)^2} + \frac{\partial^2 f}{(\partial y^3)^2} = 0$ on D. Hence Itô's formula yields

$$M_t = M_0 + \int_0^t \nabla f(B_s)\,\mathrm{d}B_s + \frac{1}{2}\int_0^t \Delta f(B_s)\,\mathrm{d}s = M_0 - \sum_{i=1}^3 \int_0^t \frac{B_s^i}{|B_s|^3}\,\mathrm{d}B_s^i.$$

Thus M is a continuous local martingale.

It remains to show that M is bounded in L^2. To this end, note that by part (b), we have

$$\begin{aligned}
E[M_t^2] &= E\left[\frac{1}{|Z + B_t|^2}\right] \\
&= \int_0^\infty \frac{1}{y}\frac{(2(t+1))^{-3/2}y^{1/2}}{\Gamma(3/2)}\exp\left(-\frac{y}{2(t+1)}\right)\mathrm{d}y \\
&= \frac{\Gamma(1/2)}{\Gamma(3/2)}\frac{1}{2(t+1)}\int_0^\infty \frac{(2(t+1))^{-1/2}y^{-1/2}}{\Gamma(1/2)}\exp\left(-\frac{y}{2(t+1)}\right)\mathrm{d}y \\
&= \frac{2\Gamma(1/2)}{\Gamma(1/2)}\frac{1}{2(t+1)} \\
&= \frac{1}{t+1}, \quad t > 0,
\end{aligned}$$

as $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$ and since we integrate the density of a $\mathrm{Gamma}(\frac{1}{2}, \frac{1}{2(t+1)})$ distribution. Therefore $\sup_{t\geqslant 0} E[M_t^2] = 1 < \infty$.

(d) For each $t > 0$,

$$\begin{aligned}
E[M_t] &= \int_0^\infty \frac{1}{\sqrt{y}}\frac{(2(t+1))^{-3/2}y^{1/2}}{\Gamma(3/2)}\exp\left(-\frac{y}{2(t+1)}\right)\mathrm{d}y \\
&= \frac{1}{\Gamma(3/2)\sqrt{2(t+1)}}\int_0^\infty (2(t+1))^{-1}\exp\left(-\frac{y}{2(t+1)}\right)\mathrm{d}y \\
&= \frac{\sqrt{2}}{\sqrt{\pi t}}.
\end{aligned}$$

In particular, the map $t \mapsto E[M_t]$ is not constant, and thus M cannot be a martingale.

Exercise 9.5 Consider a probability space $(\Omega, \mathcal{F}, P)$ supporting a Brownian motion $W = (W_t)_{t \geqslant 0}$. Denote by $\mathbb{F} = (\mathcal{F}_t)_{t \geqslant 0}$ the P-augmentation of the raw filtration generated by W. Moreover, fix $T > 0$, $a < b$, and let $F := \mathbf{1}_{\{a \leqslant W_T \leqslant b\}}$. The aim of this exercise is to find explicitly the integrand $H \in L^2_{\mathrm{loc}}(W)$ in the Itô representation

$$F = E[F] + \int_0^T H_s \, \mathrm{d}W_s. \qquad (*)$$

(a) Define the martingale $M = (M_t)_{0 \leqslant t \leqslant T}$ by $M_t := E[F \mid \mathcal{F}_t]$. Show that there exists a C^2-function $g : \mathbb{R} \times [0, T) \to \mathbb{R}$ such that

$$M_t = g(W_t, t), \quad 0 \leqslant t < T,$$

Compute g explicitly in terms of the distribution function Φ of the standard normal distribution.

(b) Let $(t_n)_{n \in \mathbb{N}}$ be a sequence of nonnegative reals with $t_n \uparrow T$. Use Itô's formula to find for each $n \in \mathbb{N}$ a predictable process H^n such that

$$M^{t_n} - M_0 = H^n \bullet W.$$

(c) Find the process $H \in L^2_{\mathrm{loc}}(W)$ on $[0, T]$ satisfying $(*)$.

Solution 9.5

(a) As $W_T - W_t \sim \mathcal{N}(0, T - t)$ is independent of $\mathcal{F}_t$, we have

$$\begin{aligned}
M_t &= P[a \leqslant W_T \leqslant b \mid \mathcal{F}_t] \\
&= P\left[\frac{a - W_t}{\sqrt{T-t}} \leqslant \frac{W_T - W_t}{\sqrt{T-t}} \leqslant \frac{b - W_t}{\sqrt{T-t}} \,\middle|\, \mathcal{F}_t\right] \\
&= P\left[\frac{W_T - W_t}{\sqrt{T-t}} \leqslant \frac{b - W_t}{\sqrt{T-t}} \,\middle|\, \mathcal{F}_t\right] - P\left[\frac{W_T - W_t}{\sqrt{T-t}} \leqslant \frac{a - W_t}{\sqrt{T-t}} \,\middle|\, \mathcal{F}_t\right] \\
&= \Phi\left(\frac{b - W_t}{\sqrt{T-t}}\right) - \Phi\left(\frac{a - W_t}{\sqrt{T-t}}\right) \\
&= g(W_t, t),
\end{aligned}$$

where

$$g(x, t) = \Phi\left(\frac{b - x}{\sqrt{T-t}}\right) - \Phi\left(\frac{a - x}{\sqrt{T-t}}\right).$$

(b) By Itô's formula and since M is a martingale, we have for $0 \leqslant t < T$ that $\mathrm{d}M_t = \frac{\partial g}{\partial x}(W_t, t) \, \mathrm{d}W_t$. We then compute, for $0 \leqslant t < T$,

$$\mathrm{d}M_t = \frac{1}{\sqrt{2\pi(T-t)}} \left(\exp\left(-\frac{(a - W_t)^2}{2(T-t)}\right) - \exp\left(-\frac{(b - W_t)^2}{2(T-t)}\right)\right) \mathrm{d}W_t.$$

In particular, we have that $M^{t_n} - M_0 = H^n \bullet W$, where

$$H_t^n := \mathbf{1}_{\{0 \leqslant t \leqslant t_n\}} \frac{1}{\sqrt{2\pi(T-t)}} \left(\exp\left(-\frac{(a-W_t)^2}{2(T-t)} \right) - \exp\left(-\frac{(b-W_t)^2}{2(T-t)} \right) \right).$$

(c) We claim that

$$H_t = \mathbf{1}_{\{0 \leqslant t < T\}} \frac{1}{\sqrt{2\pi(T-t)}} \left(\exp\left(-\frac{(a-W_t)^2}{2(T-t)} \right) - \exp\left(-\frac{(b-W_t)^2}{2(T-t)} \right) \right)$$

satisfies $(*)$. Since $P[W_T \in \mathbb{R} \setminus \{a, b\}] = 1$ and W is continuous P-a.s., we can see that $\lim_{t \uparrow T} H_t = 0$ P-a.s. So H is continuous on $[0, T]$ and hence locally bounded, so that $H \bullet W$ is well-defined and continuous on $[0, T]$. Also, we see that $M_t = g(W_t, t) \to \mathbf{1}_{\{a \leqslant W_T \leqslant b\}} = M_T$ P-a.s. as $t \uparrow T$. Now notice from the construction of H that for each $n \in \mathbb{N}$,

$$M_{t_n} - M_0 = (H^n \bullet W)_{t_n} = (H \bullet W)_{t_n}.$$

We can thus take the limit $n \to \infty$ to obtain

$$M_T - M_0 = (H \bullet W)_T.$$

Since $M_T = F$ and $M_0 = E[F]$, this implies that H satisfies $(*)$, as claimed.