Introduction to Mathematical Finance

Exercise sheet 11

Exercise 11.1 Consider a general arbitrage-free single-period market with $\mathcal{F}_0$ trivial. Fix x and let $U : (0,\infty) \to \mathbb{R}$ be a concave, increasing, continuously differentiable (utility) function such that

$$\sup_{\vartheta \in \mathcal{A}(x)} E[U(x + \vartheta \cdot \triangle X_1)] < \infty, \tag{1}$$

with

$$\mathcal{A}(x) = \{\vartheta \in \mathbb{R}^d : x + \vartheta \cdot \triangle X_1 \geq 0 \ P\text{-a.s.}, \ U(x + \vartheta \cdot \Delta X_1) \in L^1\}.$$

Furthermore, assume that the supremum is attained in an interior point ϑ^* of $\mathcal{A}(x)$.

Show that we have the *first order condition*

$$E[U'(x + \vartheta^* \cdot \triangle X_1) \triangle X_1] = 0.$$

Hint: You may use that due to concavity,

$$y \mapsto \frac{U(y) - U(z)}{y - z}, \quad y \in (0,\infty) \setminus \{z\}$$

is nonincreasing. By optimality, ϑ^* is better than $\vartheta^* + \varepsilon\eta$ for any $\eta \neq 0$ and $0 < \varepsilon \ll 1$; so take the difference of the corresponding utilities, divide by ε and look at $\varepsilon \searrow 0$. Exploit the hint to see that this quantity is monotonic in ε.

Exercise 11.2 Suppose that the utility function U is in C^2 and denote by J its conjugate. Show that $J' = -(U')^{-1}$ and J is strictly convex with $J'(0) = -\infty$, $J'(\infty) = 0$. Which assumptions on U do you use?

Exercise 11.3 Recall that $\mathcal{C}(x) := \{f \in L^0_+ : f \leq V_T \text{ for some } V \in \mathcal{V}(x)\}$ and $\mathcal{D}(z) := \{h \in L^0_+ : h \leq Z_T \text{ for some } Z \in \mathcal{Z}(z)\}$.

(a) Show that $\mathcal{C}(x)$ and $\mathcal{D}(z)$ are both convex and solid (i.e., $Y \in A$ and $Y' \leq Y$ implies $Y' \in A$).

(b) Show that $j(z) := \inf_{Z \in \mathcal{Z}(z)} E[J(Z_T)] = \inf_{h \in \mathcal{D}(z)} E[J(h)]$.

(c) Show that $E[J(Z_T)]$, for $Z \in \mathcal{Z}(z)$, is always well defined in $(-\infty, +\infty]$.