Introduction to Mathematical Finance

Exercise sheet 11

Exercise 11.1 Consider a general arbitrage-free single-period market with $\mathcal{F}_0$ trivial. Fix x and let $U : (0, \infty) \to \mathbb{R}$ be a concave, increasing, continuously differentiable (utility) function such that

$$\sup_{\vartheta \in \mathcal{A}(x)} E[U(x + \vartheta \cdot \triangle X_1)] < \infty, \tag{1}$$

with

$$\mathcal{A}(x) = \{\vartheta \in \mathbb{R}^d : x + \vartheta \cdot \triangle X_1 \geq 0 \ P\text{-a.s.}, U(x + \vartheta \cdot \Delta X_1) \in L^1\}.$$

Furthermore, assume that the supremum is attained in an interior point ϑ^* of $\mathcal{A}(x)$.

Show that we have the *first order condition*

$$E[U'(x + \vartheta^* \cdot \triangle X_1)\triangle X_1] = 0.$$

Hint: You may use that due to concavity,

$$y \mapsto \frac{U(y) - U(z)}{y - z}, \quad y \in (0, \infty) \setminus \{z\}$$

is nonincreasing. By optimality, ϑ^* is better than $\vartheta^* + \varepsilon\eta$ for any $\eta \neq 0$ and $0 < \varepsilon \ll 1$; so take the difference of the corresponding utilities, divide by ε and look at $\varepsilon \searrow 0$. Exploit the hint to see that this quantity is monotonic in ε.

Solution 11.1

Let η be any non-zero vector. Then by the assumption that ϑ^* is an interior point, $\vartheta^* + \varepsilon\eta \in \mathcal{A}(x)$ for all $0 < \varepsilon \ll 1$. Define

$$\triangle_\varepsilon^\eta = \frac{U(x + (\vartheta^* + \varepsilon\eta) \cdot \triangle X_1) - U(x + \vartheta^* \cdot \triangle X_1)}{\varepsilon},$$

for small ε as above. On $\{\eta \cdot \triangle X_1 = 0\}$, $\triangle_\varepsilon^\eta \equiv 0$, and on $\{\eta \cdot \triangle X_1 \neq 0\}$,

$$\triangle_\varepsilon^\eta = \eta \cdot \triangle X_1 \frac{U(x + (\vartheta^* + \varepsilon\eta) \cdot \triangle X_1) - U(x + \vartheta^* \cdot \triangle X_1)}{\varepsilon\eta \cdot \triangle X_1},$$

so $\triangle_\varepsilon^\eta$ is monotonically[1] increasing to $\eta \cdot \triangle X_1 U'(x + \vartheta^* \cdot \triangle X_1)$ as $\varepsilon \searrow 0$.

Note that all $\triangle_\varepsilon^\eta \in L^1(P)$, so that we can use monotone convergence. Moreover, by optimality, $E[\triangle_\varepsilon^\eta] \leq 0$ and therefore, by monotone convergence,

$$-\infty < E[\triangle_\varepsilon^\eta] \leq E[U'(x + \vartheta^* \cdot \triangle X_1)\eta \cdot \triangle X_1] = \lim_{\varepsilon \searrow 0} E[\triangle_\varepsilon^\eta] \leq 0.$$

[1] This is easily seen by splitting into two cases depending on the sign of $\eta \cdot \triangle X_1$.

Replacing η by $-\eta$ gives also ≥ 0; so $E[U'(x+\vartheta^* \cdot \triangle X_1)\eta \cdot \triangle X_1] = 0$. Finally, since η can be chosen arbitrary, we can take $\eta = e^i$ for $i = 1, ..., d$ to get

$$E[U'(x+\vartheta^* \cdot \triangle X_1)\triangle X_1] = 0.$$

Exercise 11.2 Suppose that the utility function U is in C^2 and denote by J its conjugate. Show that $J' = -(U')^{-1}$ and J is strictly convex with $J'(0) = -\infty$, $J'(\infty) = 0$. Which assumptions on U do you use?

Solution 11.2 By definition, $J(y) := \sup_{x>0}(U(x) - xy)$. So by the first order condition, using $U \in C^1$, the supremum is attained for $U'(x) - y = 0$, i.e. using that U' is continuous and strictly decreasing because U is strictly concave, at $x = (U')^{-1}(y) =: I(y)$. Therefore, we may write $J(y) = U(I(y)) - I(y)y$. Because $U \in C^2$, the RHS is continuously differentiable. So

$$\begin{aligned} J'(y) &= U'(I(y))I'(y) - I'(y)y - I(y) = yI'(y) - I'(y)y - I(y) = -I(y) \\ &= -(U')^{-1}(y). \end{aligned}$$

Clearly $J'(y) = -(U')^{-1}(y) > 0$ due to the strict concavity of U. So J' is strictly convex. Finally, $J'(0) = -\infty$, $J'(\infty) = 0$ follow from the Inada conditions on U.

Exercise 11.3 Recall that $\mathcal{C}(x) := \{f \in L^0_+ : f \leq V_T \text{ for some } V \in \mathcal{V}(x)\}$ and $\mathcal{D}(z) := \{h \in L^0_+ : h \leq Z_T \text{ for some } Z \in \mathcal{Z}(z)\}$.

(a) Show that $\mathcal{C}(x)$ and $\mathcal{D}(z)$ are both convex and solid (i.e., $Y \in A$ and $Y' \leq Y$ implies $Y' \in A$).

(b) Show that $j(z) := \inf_{Z \in \mathcal{Z}(z)} E[J(Z_T)] = \inf_{h \in \mathcal{D}(z)} E[J(h)]$.

(c) Show that $E[J(Z_T)]$, for $Z \in \mathcal{Z}(z)$, is always well defined in $(-\infty, +\infty]$.

Solution 11.3

(a) The fact that $\mathcal{C}(x)$ and $\mathcal{D}(z)$ are solid is direct from their definitions. We show that $\mathcal{D}(z)$ is convex. The argument for $\mathcal{C}(x)$ is analogous. Let $h_1, h_2 \in \mathcal{D}(z)$ with $h_1 \leq Z^1_T$ and $h_2 \leq Z^2_T$ and $Z^1, Z^2 \in \mathcal{Z}(z)$. Then for $\lambda \in (0,1)$, we have

$$\lambda h_1 + (1-\lambda) h_2 \leq \lambda Z^1_T + (1-\lambda) Z^2_T.$$

Moreover, the process $\lambda Z^1 + (1-\lambda) Z^2$ is still a nonnegative adapted process with $\lambda Z^1_0 + (1-\lambda) Z^2_0 = z$ and $(\lambda Z^1 + (1-\lambda) Z^2) V$ being a supermartingale for all $V \in \mathcal{V}(1)$. Hence $\lambda Z^1 + (1-\lambda) Z^2 \in \mathcal{Z}(z)$ and $\mathcal{D}(z)$ is convex.

(b) For "$\geq$", we just notice that $\mathcal{Z}_T(z) \subseteq \mathcal{D}(z)$. For "$\leq$", we use that J is decreasing and $h \leq Z_T$ to obtain $J(Z_T) \leq J(h)$ which implies $E[J(Z_T)] \leq E[J(h)]$. Taking suprema on both sides yields the conclusion.

(c) For any $x > 0$, $J(Z_T) \geq U(x) - x Z_T$ gives $E[Z_T] \geq U(x) - x E[Z_T]$ and $E[Z_T] \leq z$; so

$$E[J(Z_T)] \geq \sup_{x>0} \Big(U(x) - xz \Big) = J(z) > -\infty.$$