Introduction to Mathematical Finance

Exercise sheet 12

Exercise 12.1

(a) Prove the uniqueness of the solution h_z^* to the dual problem.

(b) Assuming $z \neq z'$ and $j(z), j(z') < \infty$, prove that $P[h_z^* \neq h_{z'}^*] > 0$.

Exercise 12.2

(a) Analogically to the proof of Lemma IV.5.2 show that, for fixed $0 < \mu < 1$ we can find a constant $\tilde{C} < \infty$ and $y_0 > 0$ such that

$$-J'(\mu y) < \tilde{C}\frac{J(y)}{y} \quad \text{for } 0 < y < y_0.$$

(b) Prove that if $z_n \to z$ and all z_n and z are in the interior of $\{j < \infty\}$ and $\mu_n \uparrow 1$, then

$$\lim_{n\to\infty} E[h_{z_n}^* I(\mu_n h_{z_n}^*)] = E[h_z^* I(h_z^*)].$$

Hint: Use (a) and almost repeat the proof of Lemma IV.5.3.

Exercise 12.3 Consider a general market in finite discrete time with horizon $T \in \mathbb{N}$. Let $U : (0,\infty) \to \mathbb{R}$ be an increasing and concave utility function, and denote by u the indirect utility from maximizing the utility of final wealth, i.e.,

$$u(x) = \sup_{\vartheta \in \Theta^x} E\Big[U\Big(x + G_T(\vartheta)\Big)\Big],$$

for $x > 0$, where $\Theta^x = \{\vartheta \in \Theta : \vartheta \text{ is } x\text{-admissible}\}$.

Suppose that U is strictly increasing, $U(\infty) < \infty$ and X satisfies NA. Show that if there exists an optimal strategy ϑ^* for x, then $u(x) < U(\infty)$.