ETH Zürich, SS 2025
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Introduction to Mathematical Finance

Exercise sheet 2

Exercise 2.1 Consider the one-step *binomial market* described by

$$\pi = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \begin{pmatrix} 1+r & 1+u \\ 1+r & 1+d \end{pmatrix},$$

for some $r > -1$, u and d with $u > d$.

(a) Show that this market is free of arbitrage if and only if $u > r > d$.

(b) Construct an arbitrage opportunity for a market where $u = r > d$.

Exercise 2.2 Consider the one-step *trinomial market* described by

$$\pi = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \begin{pmatrix} 1+r & 1+u \\ 1+r & 1+m \\ 1+r & 1+d \end{pmatrix},$$

for some $r > -1$, u, m and d with $u > m > d$ and $u > r > d$.

(a) Show that $\mathbb{P}(D^0)$ is convex.

(b) Calculate the set $\mathbb{P}(D^0)$ of equivalent martingale measures.

Hint: Use the probability of the 'middle outcome' as a parameter in a parametrization of $\mathbb{P}(D^0)$ as a line segment in $\mathbb{R}^3$.

Exercise 2.3

While walking around a Christmas market in Zurich, a student was offered to play the following game:

- Flip a (fair) coin
- If it comes down heads, you get CHF 2
- If tails, flip again
- If that coin comes down heads, you get the double of CHF 2
- If tails, flip again
- And so on.

To play this game, the student must pay an entrance fee η. Denote the (random) payoff of this game as X.

(a) Find the expected value of the payoff X.

(b) To decide whether to play or not, the student uses the following benchmark. He estimates his utility of owning the amount y CHF as $\ln y$. He has CHF 200 with him. If the expected utility of his capital after playing the game is greater than the utility of owning CHF 200, he accepts to play, otherwise not. So his criterion for accepting to play is:

$$\ln 200 < \sum_{n=1}^{\infty} \frac{1}{2^n} \ln(200 - \eta + 2^n).$$

Should he accept the game, if:

1. $\eta = 2$?
2. $\eta = 150$?

(c) **(*)** How much would you pay to play this game? Explain why - suggest your criteria for finding your fair entrance fee η.

(d) **(*)** Find the price η^* such that if $\eta < \eta^*$ then the student should accept to play, but if $\eta > \eta^*$ then he should refuse to play.