Introduction to Mathematical Finance

Exercise sheet 2

Exercise 2.1 Consider the one-step *binomial market* described by

$$\pi = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \begin{pmatrix} 1+r & 1+u \\ 1+r & 1+d \end{pmatrix},$$

for some $r > -1$, u and d with $u > d$.

(a) Show that this market is free of arbitrage if and only if $u > r > d$.

(b) Construct an arbitrage opportunity for a market where $u = r > d$.

Solution 2.1

(a) The market being arbitrage-free is equivalent to the existence of a probability measure $Q = (q_u, q_d)$ with

$$E_Q\left[\frac{D^1}{D^0}\right] = q_u \frac{1+u}{1+r} + q_d \frac{1+d}{1+r} = 1,$$

$q_u + q_d = 1$, $q_u > 0$, and $q_d > 0$. These equalities are satisfied by

$$q_u = \frac{r-d}{u-d} \quad \text{and} \quad q_d = \frac{u-r}{u-d},$$

which satisfy the positivity conditions if and only if $r \in (d, u)$.

(b) If $u = r$, the risky asset can only lose value, relative to the risk-free asset. An arbitrage of the first kind is therefore given by ("go long risk-free asset and short risky asset")

$$\vartheta = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This strategy costs $\vartheta \cdot \pi = 0$ at time 0 and yields

$$\mathcal{D}\vartheta = \begin{pmatrix} 0 \\ r-d \end{pmatrix}$$

at time T.

Exercise 2.2 Consider the one-step *trinomial market* described by

$$\pi = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \begin{pmatrix} 1+r & 1+u \\ 1+r & 1+m \\ 1+r & 1+d \end{pmatrix},$$

for some $r > -1$, u, m and d with $u > m > d$ and $u > r > d$.

(a) Show that $\mathbb{P}(D^0)$ is convex.

(b) Calculate the set $\mathbb{P}(D^0)$ of equivalent martingale measures.

Hint: Use the probability of the 'middle outcome' as a parameter in a parametrization of $\mathbb{P}(D^0)$ as a line segment in $\mathbb{R}^3$.

Solution 2.2

(a) Let Q^1 and Q^2 be two equivalent martingale measures and define $Q = \lambda Q^1 + (1-\lambda)Q^2$ for $\lambda \in [0,1]$. Clearly $Q[\Omega] = 1$ and $Q[\{\omega_k\}] > 0$ for all k. Furthermore,

$$E_Q\left[\frac{D^\ell}{D^0}\right] = \lambda E_{Q^1}\left[\frac{D^\ell}{D^0}\right] + (1-\lambda)E_{Q^2}\left[\frac{D^\ell}{D^0}\right] = \lambda\pi^\ell + (1-\lambda)\pi^\ell = \pi^\ell,$$

for all ℓ, showing that $Q \in \mathbb{P}(D^0)$. Since λ was arbitrary, $\mathbb{P}(D^0)$ is convex.

(b) Let Q be any probability measure on $\mathcal{F}$ and $q_i = Q[\{\omega_i\}]$ for $i \in \{u, m, d\}$. Now write down the conditions on q_i:

$$\begin{aligned} \pi^1 &= E_Q\left[\frac{D^1}{D^0}\right], \\ &= \frac{(1+u)q_u + (1+m)q_m + (1+d)q_d}{1+r}\pi^1, && \text{(Martingale property)} \\ 1 &= q_u + q_m + q_d, && (Q[\Omega] = 1) \\ q_i &\in (0,1), \quad i \in \{u, m, d\}. && (Q \approx P) \end{aligned}$$

As suggested in the hint, we parametrize this set by choosing $q_m = \lambda$. Using the two equations then yields

$$\begin{aligned} q_u &= \frac{(r-d) - (m-d)\lambda}{u-d}, \\ q_d &= \frac{(u-r) - (u-m)\lambda}{u-d}. \end{aligned}$$

Now we just have to restrict λ according to the third condition. This amounts to choosing λ such that

$$\begin{aligned} q_m \in (0,1) &\Leftrightarrow \lambda \in (0,1), \\ q_u \in (0,1) &\Leftrightarrow \lambda \in \left(\frac{r-u}{m-d}, \frac{r-d}{m-d}\right), \\ q_d \in (0,1) &\Leftrightarrow \lambda \in \left(\frac{d-r}{u-m}, \frac{u-r}{u-m}\right). \end{aligned}$$

Since $u > m > d$ and $u > r > d$ this reduces to

$$\lambda \in \left(0, \min\left\{\frac{r-d}{m-d}, \frac{u-r}{u-m}\right\}\right).$$

Hence, with the identification of $\mathbb{P}$ as a subset of $[0,1]^3$,

$$\mathbb{P} = \left\{ \left(\frac{(r-d)-(m-d)\lambda}{u-d}, \lambda, \frac{(u-r)-(u-m)\lambda}{u-d}\right) : \lambda \in \left(0, \min\left\{\frac{r-d}{m-d}, \frac{u-r}{u-m}\right\}\right) \right\}.$$

Exercise 2.3

While walking around a Christmas market in Zurich, a student was offered to play the following game:

- Flip a (fair) coin
- If it comes down heads, you get CHF 2
- If tails, flip again
- If that coin comes down heads, you get the double of CHF 2
- If tails, flip again
- And so on.

To play this game, the student must pay an entrance fee η. Denote the (random) payoff of this game as X.

(a) Find the expected value of the payoff X.

(b) To decide whether to play or not, the student uses the following benchmark. He estimates his utility of owning the amount y CHF as $\ln y$. He has CHF 200 with him. If the expected utility of his capital after playing the game is greater than the utility of owning CHF 200, he accepts to play, otherwise not. So his criterion for accepting to play is:

$$\ln 200 < \sum_{n=1}^{\infty} \frac{1}{2^n} \ln(200 - \eta + 2^n).$$

Should he accept the game, if:

1. $\eta = 2$?
2. $\eta = 150$?

(c) How much would you pay to play this game? Explain why - suggest your criteria for finding your fair entrance fee η.

(d) * Find the price η^* such that if $\eta < \eta^*$ then the student should accept to play, but if $\eta > \eta^*$ then he should refuse to play.

Solution 2.3

(a) $\mathrm{E}[X] = \sum_{n=1}^{\infty} 2^n \cdot \frac{1}{2^n} = +\infty.$

(b.1) If $\eta = 2$, then the expected utility is

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \ln(198 + 2^n) > \sum_{n=1}^{\infty} \frac{1}{2^n} \ln 200 = \ln 200.$$

Thus, the student should accept to play.

(b.2) If $\eta = 150$, then the expected utility is

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \ln(50 + 2^n) < \sum_{n=1}^{\infty} \frac{1}{2^n} (\ln 50 + \ln(2^n)) = \ln 50 + \ln 2 \sum_{n=1}^{\infty} \frac{n}{2^n}$$
$$= \ln 50 + 2 \ln 2 = \ln 200.$$

In the last calculation we used that $50 + 2^n < 50 \cdot 2^n$. Hence

$$\ln(50 + 2^n) < \ln(50 \cdot 2^n) = \ln 50 + n \ln 2.$$

Thus, the student should refuse to play.

(d) We define the function:

$$F(\eta) = \sum_{n=1}^{\infty} \frac{1}{2^n} \ln(200 - \eta + 2^n)$$

We need to find η^* such that:

$$F(\eta^*) = \ln(200)$$

Since we know that $F(5) > \ln(200)$ and $F(150) < \ln(200)$, we apply the bisection method to numerically approximate η^*. The steps are:

1. Set $a = 5$ and $b = 150$, and compute $F(a)$ and $F(b)$.
2. Compute the midpoint $c = \frac{a+b}{2}$ and evaluate $F(c)$.
3. If $F(c) > \ln(200)$, set $a = c$, else set $b = c$.
4. Repeat (2) and (3) until convergence, i.e., until $|b - a|$ is sufficiently small.

Using numerical computation, we iteratively apply the bisection method and obtain:

$$\eta^* \approx 8.72$$

Thus, the student should accept the game if $\eta < 8.72$ and refuse if $\eta > 8.72$.