Introduction to Mathematical Finance

Exercise sheet 3

Exercise 3.1 Let H be a payoff at time T and Ψ a consistent price system.

(a) Show that if H is attainable, then $R^H \in \mathrm{Span}(R^{D^\ell}, 0 \leq \ell \leq N)$.

(b) Suppose that Q is the EMM associated to Ψ and H is attainable. Also assume that D^0 is a bond with interest rate r. Compute $E_Q\left[R^H\right]$.

Exercise 3.2 Recall the setup in Exercise 2.2, where

$$\pi = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \begin{pmatrix} 1+r & 1+u \\ 1+r & 1+m \\ 1+r & 1+d \end{pmatrix}$$

for some $r > -1$, u, m and d with $u \geq m \geq d$ and $u > r > d$. Denote by $\mathbb{P}_a$ the set of all martingale measures Q which are absolutely continuous with respect to P, i.e., $Q \ll P$.

(a) Show that $\mathbb{P}_a = \overline{\mathbb{P}}$. Here we identity $\mathbb{P}$ with a subset of $\mathbb{R}_+^K = \mathbb{R}_+^3$ and denote by $\overline{}$ the closure in $\mathbb{R}^K$.
Hint: Exercise 2.2

(b) Use (a) to show that for any random variable X,

$$\sup_{Q \in \mathbb{P}} E_Q[X] = \sup_{Q \in \mathbb{P}_a} E_Q[X].$$

(c) Show that for any payoff H, the supremum

$$\sup_{Q \in \mathbb{P}_a} E_Q\left[\frac{H}{D^0}\right]$$

is attained in some $Q \in \mathbb{P}_a$. Does this imply that the market is complete?

Exercise 3.3 Let

$$\pi = \begin{pmatrix} 1 \\ 1000 \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \begin{pmatrix} 1.1 & 1200 \\ 1.1 & 1100 \\ 1.1 & 800 \end{pmatrix}.$$

This is similar to the example with the gold market from the lecture, but now with three possible outcomes. Denote by H the payoff of a put option with strike $K = 900$, i.e.,

$$H = (900 - D^1)^+ = \begin{pmatrix} 0 \\ 0 \\ 100 \end{pmatrix}.$$

(a) Find

$$\sup_{Q \in \mathbb{P}(D^0)} E_Q\left[\frac{H}{D^0}\right].$$

(b) Find

$$\inf\{\vartheta \cdot \pi : \mathcal{D}\vartheta \geq H\}.$$

(c) Construct a market with $\mathbb{P}_a \neq \overline{\mathbb{P}}$, where we use the notation from Exercise 3.2.