Introduction to Mathematical Finance

Exercise sheet 3

Exercise 3.1 Let H be a payoff at time T and Ψ a consistent price system.

(a) Show that if H is attainable, then $R^H \in \operatorname{Span}(R^{D^\ell}, 0 \leq \ell \leq N)$.

(b) Suppose that Q is the EMM associated to Ψ and H is attainable. Also assume that D^0 is a bond with interest rate r. Compute $E_Q\left[R^H\right]$.

Solution 3.1

(a) Since H is attainable, there exists some $\vartheta \in \mathbb{R}^N$ such that $H = \mathcal{D}\vartheta$. We thus have

$$R^H = \frac{H}{\Psi(0,H)} - 1 = \frac{\mathcal{D}\vartheta - \vartheta \cdot \pi}{\vartheta \cdot \pi} = \sum_{\ell=0}^{N} \frac{\vartheta^\ell (D^\ell - \pi^\ell)}{\pi^\ell} \frac{\pi^\ell}{\vartheta \cdot \pi} = \sum_{\ell=0}^{N} \frac{\vartheta^\ell \pi^\ell}{\vartheta \cdot \pi} R^{D^\ell}.$$

where in the first equality we have used the definition of the return of H.

(b) By Theorem I.5.2, we know that there exists a bijection between set $\mathbb{P}$ of all EMMs Q and the set of all consistent price systems Ψ on $\mathcal{C}$ given by $E_Q[H] = \Psi(0, D^0 H) = (1+r)\Psi(0,H)$ (note that in the last equality we have used the linearity of the price systems Ψ and the fact that D^0 is a bond with interest rate r). Moreover since $R^H = \frac{H - \Psi(0,H)}{\Psi(0,H)}$, we have

$$E_Q\left[R^H\right] = \frac{E_Q[H] - \Psi(0,H)}{\Psi(0,H)} = r.$$

Exercise 3.2 Recall the setup in Exercise 2.2, where

$$\pi = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \begin{pmatrix} 1+r & 1+u \\ 1+r & 1+m \\ 1+r & 1+d \end{pmatrix}$$

for some $r > -1$, u, m and d with $u \geq m \geq d$ and $u > r > d$. Denote by $\mathbb{P}_a$ the set of all martingale measures Q which are absolutely continuous with respect to P, i.e., $Q \ll P$.

(a) Show that $\mathbb{P}_a = \overline{\mathbb{P}}$. Here we identity $\mathbb{P}$ with a subset of $\mathbb{R}_+^K = \mathbb{R}_+^3$ and denote by $\overline{}$ the closure in $\mathbb{R}^K$.
Hint: Exercise 2.2

(b) Use (a) to show that for any random variable X,

$$\sup_{Q \in \mathbb{P}} E_Q[X] = \sup_{Q \in \mathbb{P}_a} E_Q[X].$$

(c) Show that for any payoff H, the supremum

$$\sup_{Q \in \mathbb{P}_a} E_Q\left[\frac{H}{D^0}\right]$$

is attained in some $Q \in \mathbb{P}_a$. Does this imply that the market is complete?

Solution 3.2

(a) Recall from the solution of Exercise 2.2 that

$$\mathbb{P}_a = \left\{ \left(\frac{(r-d)-(m-d)\lambda}{u-d}, \lambda, \frac{(u-r)-(u-m)\lambda}{u-d} \right) : \lambda \in \left[0, \min\left\{\frac{r-d}{m-d}, \frac{u-r}{u-m}\right\}\right] \right\},$$

which is equal to $\overline{\mathbb{P}}$.

Alternative solution to (a). Because $u > r > d$, $\mathbb{P} \neq \varnothing$. Take $P^* \in \mathbb{P}$ and any $Q \in \mathbb{P}_a$. Consider $Q_\varepsilon := \varepsilon P^* + (1-\varepsilon)Q$ and observe that

- Q_ε is a martingale measure since both P^* and Q are martingale measures
- Q_ε and P are equivalent since $P^* \sim P$ and $Q << P$

Taking the limit as $\epsilon \to 0$, we get that $Q \in \overline{\mathbb{P}}$.

(b) Let R be an element in $\mathbb{P}_a$. Let Q be an arbitrary element in $\mathbb{P}$ and construct $Q^\varepsilon = \varepsilon Q + (1-\varepsilon)R$. Then $Q^\varepsilon \in \mathbb{P}$ for all $\varepsilon \in (0,1]$ by construction, and

$$\lim_{\varepsilon \searrow 0} E_{Q^\varepsilon}[X] = \lim_{\varepsilon \searrow 0} \varepsilon E_Q[X] + (1-\varepsilon)E_R[X] = E_R[X],$$

implying that

$$\sup_{Q\in\mathbb{P}} E_Q[X] \geq \sup_{Q\in\mathbb{P}_a} E_Q[X].$$

The converse inequality inequality is trivial.

(c) By (a), $\mathbb{P}_a$ is closed and bounded, so $\mathbb{P}_a$ is compact. This uses again that we identify $\mathbb{P}$ and $\mathbb{P}_a$ with subset of $\mathbb{R}^K_+$. Also the mapping

$$\begin{aligned} \ell : \mathbb{P}_a &\to \mathbb{R}, \\ Q &\mapsto E_Q\left[X\right] \end{aligned}$$

is linear and hence continuous because $\mathbb{P}_a$ is finite-dimensional here. Thus the supremum is in fact attained.

Clearly the above result does not depend on the particular values of r, u, m, d. All we need is $u > r > d$ to have $\mathbb{P} \neq \varnothing$. So we can adjust $\mathcal{D}$ to obtain an incomplete market by taking $u > m > d$, while the conclusion of (c) is still true.

Exercise 3.3 Let

$$\pi = \begin{pmatrix} 1 \\ 1000 \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \begin{pmatrix} 1.1 & 1200 \\ 1.1 & 1100 \\ 1.1 & 800 \end{pmatrix}.$$

This is similar to the example with the gold market from the lecture, but now with three possible outcomes. Denote by H the payoff of a put option with strike $K = 900$, i.e.,

$$H = (900 - D^1)^+ = \begin{pmatrix} 0 \\ 0 \\ 100 \end{pmatrix}.$$

(a) Find

$$\sup_{Q \in \mathbb{P}(D^0)} E_Q\left[\frac{H}{D^0}\right].$$

(b) Find

$$\inf\{\vartheta \cdot \pi : \mathcal{D}\vartheta \geq H\}.$$

(c) Construct a market with $\mathbb{P}_a \neq \overline{\mathbb{P}}$, where we use the notation from Exercise 3.2.

Solution 3.3 Note that $\mathcal{D}$ is of the form

$$\mathcal{D} = \begin{pmatrix} \pi^0(1+r) & \pi^1(1+u) \\ \pi^0(1+r) & \pi^1(1+m) \\ \pi^0(1+r) & \pi^1(1+d) \end{pmatrix},$$

with $r = 0.1$, $u = 0.2$, $m = 0.1$ and $d = -0.2$.

(a) From Exercise 2.2 we know that

$$\mathbb{P} = \left\{ \left(\frac{(r-d) - (m-d)\lambda}{u-d}, \lambda, \frac{(u-r) - (u-m)\lambda}{u-d} \right) : \lambda \in \left(0, \min\left\{\frac{r-d}{m-d}, \frac{u-r}{u-m}\right\}\right) \right\}.$$

Set $q_i = Q\left[\{\omega_i\}\right]$ for $i \in \{1,2,3\}$. Since $E_Q\left[H/D^0\right] = \sum_{i=1}^3 q_i H(\omega_i)/D^0 = \frac{100 q_3}{1.1}$, which is decreasing in λ, we find the supremum by setting $\lambda = 0$ to obtain

$$\sup_{Q \in \mathbb{P}} E_Q\left[\frac{H}{D^0}\right] = \frac{100}{1.1}\frac{1}{4} = \frac{25}{1.1} = 22.73.$$

(b) Writing down the condition $\mathcal{D}\vartheta \geq H$, we obtain the optimisation problem

$$\begin{aligned} \min \quad & \vartheta^0 + \pi^1\vartheta^1 \\ \text{s.t.} \quad & 11\vartheta^0 + 12\pi^1\vartheta^1 \geq 0, \\ & \vartheta^0 + \pi^1\vartheta^1 \geq 0, \\ & 11\vartheta^0 + 8\pi^1\vartheta^1 \geq 10H_3 = 1000. \end{aligned}$$

Note that the solution is found at an extreme point of the set of feasible solutions, and that the second condition cannot be satisfied with equality without violating the other two. Therefore, solving the outer two inequalities with equality gives

$$\vartheta^s = \begin{pmatrix} \frac{30}{11} \\ -\frac{5}{2\pi^1} \end{pmatrix} H_3$$

as a solution to the optimisation problem, and

$$\pi \cdot \vartheta^s = \frac{25}{1.1}.$$

(c) Extend the market with the asset H at the price $\pi_s(H)$. Denote by $\tilde{\mathbb{P}}$ (resp. $\tilde{\mathbb{P}}_a$) the set of all equivalent (resp. all absolutely continuous) martingale measures with the numéraire D^0 in the extended market. Then, from the characterisation of martingale measures to the original market, we conclude that

$$\tilde{\mathbb{P}}_a = \left\{ \left(\frac{3}{4}, 0, \frac{1}{4} \right) \right\},$$

(this is the measure corresponding to $\lambda = 0$) which is of course not an *equivalent* martingale measure. Hence, $\tilde{\mathbb{P}} = \varnothing$ and $\tilde{\mathbb{P}}_a \neq \overline{\tilde{\mathbb{P}}}$.

A simpler example is given by the market

$$\pi = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

where the only measure satisfying the martingale property is identified by $(0, 1)$, which is not equivalent to P.