Introduction to Mathematical Finance

Exercise sheet 4

Exercise 4.1 Let $(\mathcal{D}, \pi)$ be an arbitrage-free market with numéraire. You can assume that in such a market, for any payoff H, there exists a strategy ϑ^s which attains the infimum in the definition of $\pi_s(H)$.

Consider a payoff H which is not attainable in $\mathcal{D}$ and $\pi_s(H)$ the seller's price for H, i.e.,

$$\pi_s(H) = \inf\{\vartheta \cdot \pi : \vartheta \in \mathbb{R}^N \text{ with } \mathcal{D}\vartheta \geq H\}.$$

Denote by $(\mathcal{D}^e, \pi^e)$ the extended market $(\mathcal{D}, H, \pi, \pi_s(H))$.

(a) Show that $(\mathcal{D}^e, \pi^e)$ always admits an arbitrage opportunity of the first kind.

(b) Show that $(\mathcal{D}^e, \pi^e)$ does not admit an arbitrage opportunity of the second kind.

Exercise 4.2 Let H be a payoff at time T. Assume the binomial model (Exercise 2.1) with $d < r < u$. Suppose that $H = f(D^1)$ for some convex function $f \geq 0$. Compute $\pi_s(H)$.

Exercise 4.3 Consider an arbitrage-free market with a single risky asset D^1. Assume D^0 is a bond with interest rate $r > -1$. Set

$$\pi = \begin{pmatrix} 1 \\ \pi^1 \end{pmatrix}.$$

Recall that a *call option* on D^1 with strike K is defined by $H^c := (D^1 - K)^+$ and a put option with strike K is defined by $H^p := (K - D^1)^+$.

Suppose that the market is complete. Show that the arbitrage-free prices $\pi(H^c)$ and $\pi(H^p)$ of H^c and H^p, respectively, are related by

$$\pi(H^c) - \pi(H^p) = \pi^1 - \frac{K}{1+r}.$$

This relation is known as the *put-call parity*.