Introduction to Mathematical Finance

Exercise sheet 4

Exercise 4.1 Let $(\mathcal{D}, \pi)$ be an arbitrage-free market with numéraire. You can assume that in such a market, for any payoff H, there exists a strategy ϑ^s which attains the infimum in the definition of $\pi_s(H)$.

Consider a payoff H which is not attainable in $\mathcal{D}$ and $\pi_s(H)$ the seller's price for H, i.e.,

$$\pi_s(H) = \inf\{\vartheta \cdot \pi : \vartheta \in \mathbb{R}^N \text{ with } \mathcal{D}\vartheta \geq H\}.$$

Denote by $(\mathcal{D}^e, \pi^e)$ the extended market $(\mathcal{D}, H, \pi, \pi_s(H))$.

(a) Show that $(\mathcal{D}^e, \pi^e)$ always admits an arbitrage opportunity of the first kind.

(b) Show that $(\mathcal{D}^e, \pi^e)$ does not admit an arbitrage opportunity of the second kind.

Solution 4.1

(a) Denote by ϑ^s the strategy in $(\mathcal{D}, \pi)$ which attains the infimum in the definition of $\pi_s(H)$. Consider the strategy in $(\mathcal{D}^e, \pi^e)$ given by

$$\vartheta^a = \begin{pmatrix} \vartheta^s \\ -1 \end{pmatrix}.$$

Then, by construction of ϑ^s,

$$\vartheta^a \cdot \pi^e = \vartheta^s \cdot \pi - \pi_s(H) = 0$$

and

$$\mathcal{D}^e \vartheta^a = \mathcal{D}\vartheta^s - H \geq 0,$$

with strict inequality in some outcome (or else H would be attainable), showing that ϑ^a is an arbitrage opportunity of the first kind in (D^e, π^e).

(b) Let $N + 1$ be the index of the asset with payoff H in the extended market and denote again by ϑ^s the strategy in $(\mathcal{D}, \pi)$ which attains the infimum in the definition of $\pi_s(H)$. Suppose there exists an arbitrage opportunity ϑ^a of the second kind in $(\mathcal{D}^e, \pi^e)$. Denote by ϑ^{a-} the vector

$$\begin{pmatrix} \vartheta^a_1 \\ \vdots \\ \vartheta^a_N \end{pmatrix}.$$

Define ϑ according to

$$\vartheta = \vartheta^{a-} + \vartheta^a_{N+1}\vartheta^s.$$

Then

$$\begin{aligned}\vartheta^a \cdot \pi^e &= \vartheta^{a-} \cdot \pi + \vartheta^a_{N+1}\pi_s(H) \\ &= \vartheta^{a-} \cdot \pi + \vartheta^a_{N+1}\vartheta^s \cdot \pi \\ &= \vartheta \cdot \pi\end{aligned}$$

In the second equality, we used that ϑ^s the strategy in $(\mathcal{D}, \pi)$ from which attains the infimum in the definition of $\pi_s(H)$ and the last equality comes from the definition of ϑ. Since by assumption ϑ^a is an arbitrage of the second kind in (D^e, π^e), we have $\vartheta \cdot \pi < 0$.

If $\vartheta^a_{N+1} \geq 0$, then

$$\begin{aligned}\mathcal{D}\vartheta &= \mathcal{D}\vartheta^{a-} + \vartheta^a_{N+1}\mathcal{D}\vartheta^s \\ &\geq \mathcal{D}\vartheta^{a-} + \vartheta^a_{N+1}H \\ &= \mathcal{D}^e\vartheta^a \geq 0\end{aligned}$$

The first equality comes from the definition of ϑ; in the second line we used that ϑ^s the strategy in $(\mathcal{D}, \pi)$ from which attains the infimum in the definition of $\pi_s(H)$ and so in particular $\mathcal{D}\vartheta^s \geq H$; the last equality is a consequence of the definition of ϑ^{a-}. This implies that $(\mathcal{D}, \pi)$ has an arbitrage of the second kind. However, since that market is arbitrage-free, we reach a contradiction.

If, on the other hand, $\vartheta^a_{N+1} < 0$, then

$$\begin{aligned}0 \leq \mathcal{D}^e\vartheta^a &= \mathcal{D}\vartheta^{a-} + \vartheta^a_{N+1}H \\ &= \mathcal{D}\vartheta^{a-} - |\vartheta^a_{N+1}|H\end{aligned}$$

The first inequality holds because ϑ^a is an arbitrage of the second kind in (D^e, π^e); the equality on the first line follows from the definition of $\mathcal{D}^e$ and ϑ^a; and the equality of the second line uses the assumption $\vartheta^a_{N+1} < 0$. This implies that

$$|\vartheta^a_{N+1}|H \leq \mathcal{D}\vartheta^{a-}, \text{ hence } H \leq \mathcal{D}\frac{\vartheta^{a-}}{|\vartheta^a_{N+1}|}.$$

Thus,

$$\vartheta^a \cdot \pi^e = |\vartheta^a_{N+1}|\left(\frac{\vartheta^{a-}}{|\vartheta^a_{N+1}|} \cdot \pi - \pi_s(H)\right) \geq 0$$

by the definition of $\pi_s(H)$. This contradicts the fact that ϑ^a is an arbitrage of the second kind in $(\mathcal{D}^e, \pi^e)$.

Conclusion: We must have $\vartheta^a_{N+1} \geq 0$ but in that case we have find a contradiction so there cannot exist an arbitrage of the second kind in (D^e, π^e).

Exercise 4.2 Let H be a payoff at time T. Assume the binomial model (Exercise 2.1) with $d < r < u$. Suppose that $H = f(D^1)$ for some convex function $f \geq 0$. Compute $\pi_s(H)$.

Solution 4.2 By Exercise 2.1, a binomial market with $d < r < u$ is arbitrage free and complete. Indeed in Exercise 2.1 we found a unique EMM Q (under the assumption $d < r < u$) given by

$$q_u = \frac{r-d}{u-d}, \quad q_d = \frac{u-r}{u-d}$$

and so

- by the First Fundamental Theorem of Asset Pricing (Theorem I.4.3), the market is arbitrage-free (since $\mathbb{P} \neq \emptyset$)
- by the Second Fundamental Theorem of Asset Pricing (Theorem I.4.5), the market is complete (since $|\mathbb{P}| = 1$)

We use Theorem I.7.2 to compute $\pi_s(H)$:

$$\pi_s(H) = \sup_{Q \in \mathbb{P}} E_Q\left[\frac{H}{D^0}\right] = E_Q\left[\frac{H}{1+r}\right] = \frac{f(1+u)}{1+r}\frac{r-d}{u-d} + \frac{f(1+d)}{1+r}\frac{u-r}{u-d}.$$

Exercise 4.3 Consider an arbitrage-free market with a single risky asset D^1. Assume D^0 is a bond with interest rate $r > -1$. Set

$$\pi = \begin{pmatrix} 1 \\ \pi^1 \end{pmatrix}.$$

Recall that a *call option* on D^1 with strike K is defined by $H^c := (D^1 - K)^+$ and a put option with strike K is defined by $H^p := (K - D^1)^+$.

Suppose that the market is complete. Show that the arbitrage-free prices $\pi(H^c)$ and $\pi(H^p)$ of H^c and H^p, respectively, are related by

$$\pi(H^c) - \pi(H^p) = \pi^1 - \frac{K}{1+r}.$$

This relation is known as the *put-call parity*.

Solution 4.3 1st solution: using EMM

Since the market is complete, there is a unique EMM Q. We observe $H^c - H^p = D^1 - K$. Then we discount by D^0 and take expectation under Q to obtain

$$\pi(H^c) - \pi(H^p) = E_Q\left[\frac{D^1}{D^0}\right] - E_Q\left[\frac{K}{D^0}\right] = \pi^1 - \frac{K}{1+r}$$

2nd solution: using a replication argument

The idea is to find the initial price of the put option by finding an investment strategy consisting of investments in the bond D^0, the risky asset D^1 and the call option C, that replicates the payoff of the put option. In the same way as in the first solution, we have $H^p = K - D^1 + H^c$ and thus one can easily see that the strategy consisting of

- being long K units of bond
- being short one unit of D^1
- being long one unit of C

replicates the terminal payoff of the put option. By no arbitrage, the initial value of the put option must coincide with the initial value of our replicating portfolio and hence

$$\pi(H^p) = \frac{K}{1+r} - \pi^1 + \pi(H^c)$$