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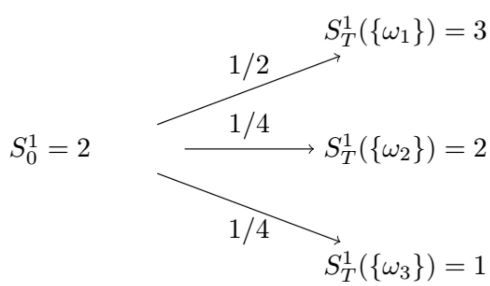
Introduction to Mathematical Finance

Exercise sheet 5

Exercise 5.1 Consider a trinomial two-asset model. The first asset is a risk-free bond with initial value $S_0^0 = 1$ and the second asset is a risky stock with initial value $S_0^1 = 2$ and whose evolution under the real world measure P is given by the following tree:



We also suppose that the spot interest rate is $r = 0$.

(a) Find all risk-neutral measures for this model.

Now introduce a call option on the risky asset with strike $K = 2$ and maturity T.

(b) What is the terminal payoff H of this contingent claim?

(c) Find the least expensive super replicating portfolio, i.e. the portfolio that attains the infimum in the definition of $\pi_s(H)$.

(d) Find the most expensive sub-replicating portfolio.

Solution 5.1

(a) An equivalent martingale measure solves $E_Q[S_1^1] = S_0^1$ (no discounting is needed since $r = 0$). By Lemma I.4.1, we can identify the measure Q with a vector $q = (q_1, q_2, q_3)^\top \in \mathbb{R}^3_{++}$ where $q_1 = Q(S_1^1 = 3)$, $q_1 = Q(S_1^1 = 2)$ and $q_1 = Q(S_1^1 = 1)$. Q being an EMM, give the following equations:

$$\begin{cases} 3q_1 + 2q_2 + q_3 = 2 \\ q_1 + q_2 + q_3 = 1 \\ q_1, q_2, q_2 > 0 \end{cases}$$

This gives that the set of all EMMs is given by

$$\mathbb{P} = \{(q_1, 1-2q_1, q_1) \quad | \quad 0 < q_1 < 1/2\}$$

(b) The terminal payoff of the call option is $(S_T - K)^+$.

(c) Denoting θ^0 and θ^1 the holding in the bond and the stock, to super-replicate the payout H, we are asked to minimize $\theta^0 + 2\theta^1$ subject to the constraints

$$\begin{cases} \theta^0 + 3\theta^1 \geq 1 \\ \theta^0 + 2\theta^1 \geq 0 \\ \theta^0 + \theta^1 \geq 0 \end{cases}$$

By Theorem I.7.2 and the bonus question of Problem 1, we now that there exist a vector $\theta^* = (\theta^{0^*}, \theta^{1^*})^\top$ such that

$$\theta^{0^*} S_0^0 + \theta^{1^*} S_0^1 = \sup_{Q \in \mathbb{P}} E_Q[H] = \sup_{0<q_1<1/2} [q_1 + (1-2q_1) \cdot 0 + q_1 \cdot 0] = 1/2$$

Unfortunately, the Linear Programming duality principle only tells us how to calculate the seller's price but tells nothing about the strategy that would give that price. However, for this problem, we can easily see that $\theta^* = (-1/2, 1/2)^\top$.

(d) Similarly, the most expensive sub-replication cost is 0 and is attained for $\theta^* = (0,0)^\top$.

Exercise 5.2

Consider a model with $d = 1$ traded risky asset X with $X_0 = 1$ and

$$\Delta X_k = \eta_k, \qquad k = 1, 2, 3,$$

where the η_k are i.i.d. $\eta_1 \sim \mathcal{N}(0, 1)$-distributed.

(a) Suppose that a trader decides at time $k = 0$ to buy 2 shares, to sell 3 shares at $k = 1$ and then to buy 1 share at time $k = 2$. Denote by G_k his cumulative gain from the corresponding self-financing trading strategy. Find the distribution of G_3.

(b) Suppose that $\mathcal{F}_k = \sigma(X_1, \dots, X_k)$ for $k = 1, 2, 3$. Show that there is no arbitrage in this model.

Solution 5.2

(a) Recall the following property of the Gaussian distribution: if $Y \sim \mathcal{N}(a, b)$, $Z \sim \mathcal{N}(c, d)$ and Y, Z are independent, then for any $\alpha, \beta \in \mathbb{R}$, we have $\alpha Y + \beta Z \sim \mathcal{N}(\alpha a + \beta c, \alpha^2 b + \beta^2 d)$. Thus $G_3 = 2\Delta X_1 - 3\Delta X_2 + \Delta X_3 \sim \mathcal{N}(0, 14)$.

(b) From the definition of the model, we have $E[(X_k - X_{k-1})|\mathcal{F}_{k-1}] = 0, =$ so that X is a martingale. Hence by Proposition II.2.3, there is no arbitrage.

Exercise 5.3 Consider the standard model for a financial market in finite discrete time with a numéraire S^0.

(a) Show that a strategy ψ is self-financing for S if and only if it is self-financing for S/S^0.

(b) Show that S satisfies NA' if and only if S/S^0 satisfies NA'.

(c) Show that S satisfies NA if and only if S/S^0 satisfies NA.

Solution 5.3

(a) Notice that for $k = 1, \dots, T-1$

$$(\psi_{k+1} - \psi_k)S_k = 0 \text{ if and only if } (\psi_{k+1} - \psi_k)S_k/S_k^0 = 0.$$

That means that a strategy ψ is self-financing for S if and only if it is self-financing for S/S^0.

(b) We know that for a numéraire $S^0 > 0$ we have $V(\psi) = \widetilde{V}(\psi)/S^0$, so that $\{-\widetilde{V}_0(\psi) \geq 0 \text{ a.s.}, \widetilde{V}_T(\psi) \geq 0 \text{ a.s.}\}$ if and only if $\{-V_0(\psi) \geq 0 \text{ a.s.}, V_T(\psi) \geq 0 \text{ a.s.}\}$ Thus we conclude that existence of (generalized) arbitrage for S is equivalent to the existence of arbitrage for S/S^0.

(c) **Claim.** NA for $S \Leftrightarrow$ NA' for S.

Clear that NA' for $S/S^0 \Rightarrow$ NA for S/S^0. To show the converse, we will use the following strategy: NA for $S/S^0 \Rightarrow$ NA for $S \Rightarrow$ NA' for $S \Rightarrow$ NA' for S/S^0. The only implication here to prove is "NA for $S/S^0 \Rightarrow$ NA for S".

Suppose that NA for S/S^0 holds. Consider the construction in "4) $\Rightarrow$ 5)" of P II.2.1. This constructs $\bar{\psi}$, self-financing, with $V_0(\bar{\psi}) = 0$ P-a.s, $V_T(\bar{\psi}) \in L^0_+ \setminus \{0\}$, and $V(\bar{\psi}) \geq 0$ i.e. an arbitrage opportunity of first kind for S/S^0 which is 0-admissible for S/S^0.

Multiplying everywhere by $S^0 > 0$ gives $\widetilde{V}_0(\bar{\psi}) = 0$ P-a.s., $\widetilde{V}_T(\bar{\psi}) \in L^0_+ \setminus \{0\}$ and $\widetilde{V}(\bar{\psi}) \geq 0$, i.e., an arbitrage of first kind for S which is 0-admissible for S. This is exactly what we want.