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Introduction to Mathematical Finance

Exercise sheet 6

Exercise 6.1 Consider a probability space $(\Omega, \mathcal{F}, P)$ with $\mathcal{F} = \sigma(A_1, \ldots, A_n)$, where $\bigcup_{i=1}^n A_i = \Omega$ and $A_i \bigcap A_j = \emptyset$ for $i \neq j$. A probability measure Q on $\mathcal{F}$ is called absolutely continuous with respect to P if for any $A \in \mathcal{F}$, $P[A] = 0$ implies that $Q[A] = 0$.

(a) Show directly, without using the Radon–Nikodym theorem, that Q is absolutely continuous with respect to P if and only if there exists a random variable $\xi \geq 0$ with $E^P[\xi] = 1$ and

$$Q[A] = \int_A \xi dP \quad \text{for all } A \in \mathcal{F}.$$

(b) Two probability measures Q and P on $\mathcal{F}$ are equivalent on $\mathcal{F}$ if for any $A \in \mathcal{F}$, we have $Q(A) = 0$ if and only if $P[A] = 0$. Construct an example where Q is absolutely continuous with respect to P, but Q and P are not equivalent.

Exercise 6.2

(a) Suppose that $(\xi_k)_{k\in\mathbb{N}}$ are independent integrable random variables with expectation 1. Define the process $X = \{X_n\}_{n\in\mathbb{N}_0}$ by $X_n := \prod_{k=1}^{n} \xi_k$. Show that X is a martingale for its natural filtration.

(b) Give an example of a stochastic process in discrete time which is not locally bounded.

Exercise 6.3

Consider a sequence $(\xi_k)_{k\in\mathbb{N}}$ of i.i.d. random variables with $\xi_1 \sim \mathcal{N}(0,1)$. Define the process $M = (M_n)_{n\in\mathbb{N}_0}$ by $M_n := \sum_{k=1}^n \xi_k$. Let $\mathbb{F} = (\mathcal{F}_k)_{k\in\mathbb{N}_0}$ be the natural filtration of M.

(a) Show that $X_n := M_n^2 - n, n \in \mathbb{N}_0$, is a martingale.

(b) Show that $Y_n := \exp(M_n - n/2), n \in \mathbb{N}_0$, is a martingale.

(c) For any bounded predictable process $\alpha = (\alpha_i)_{i\in\mathbb{N}}$ define $N := \alpha \cdot M$ so that $N_k = \sum_{i=1}^{k} \alpha_i(M_i - M_{i-1})$ for $k \in \mathbb{N}_0$. Define also $\langle N \rangle = (\langle N \rangle_k)_{k\in\mathbb{N}_0}$ by $\langle N \rangle_k := \sum_{i=1}^{k} \alpha_i^2$. Show that $X := N^2 - \langle N \rangle$ and $Y := \exp(N - \langle N \rangle/2)$ are martingales.

Exercise 6.4

Using the notions from the lecture, show that the following are equivalent:

(a) $S = S^0(1, X)$ satisfies NA.

(b) $\mathcal{G}_{adm} \bigcap L^0_+ = \{0\}$.

(c) $\mathcal{C}_{adm} \bigcap L^0_+ = \{0\}$.