# Introduction to Mathematical Finance

## Exercise sheet 6

**Exercise 6.1** Consider a probability space $(\Omega, \mathcal{F}, P)$ with $\mathcal{F} = \sigma(A_1, \ldots, A_n)$, where $\bigcup_{i=1}^n A_i = \Omega$ and $A_i \bigcap A_j = \emptyset$ for $i \neq j$. A probability measure $Q$ on $\mathcal{F}$ is called absolutely continuous with respect to $P$ if for any $A \in \mathcal{F}$, $P[A] = 0$ implies that $Q[A] = 0$.

(a) Show directly, without using the Radon–Nikodym theorem, that $Q$ is absolutely continuous with respect to $P$ if and only if there exists a random variable $\xi \geq 0$ with $E^P[\xi] = 1$ and

$$Q[A] = \int_A \xi dP \quad \text{for all } A \in \mathcal{F}.$$

(b) Two probability measures $Q$ and $P$ on $\mathcal{F}$ are equivalent on $\mathcal{F}$ if for any $A \in \mathcal{F}$, we have $Q(A) = 0$ if and only if $P[A] = 0$. Construct an example where $Q$ is absolutely continuous with respect to $P$, but $Q$ and $P$ are not equivalent.

**Solution 6.1**

(a) Consider first $Q$ defined by $Q[A] = \int_A \xi dP$ for $A \in \mathcal{F}$. From the definition of $\mathcal{F}$ it follows that any set $A \in \mathcal{F}$ is of the form $A = \bigcup_{j \in J} A_j$, where $J \subseteq \{1, ..., n\}$. So that for any $A \in \mathcal{F}$ $Q[A] = \sum_{j \in J} Q[A_j]$. From $\xi \geq 0$ and $E^P[\xi] = 1$ it is clear that $Q$ is a probability measure on $\mathcal{F}$. Since $\xi$ is a random variable on $(\Omega, \mathcal{F})$, it is of the form $\xi = \sum_{i=1}^n c_i I_{A_i}$ for some $c_i \geq 0$. Since for any $A \in \mathcal{F}$ $Q[A] = \sum_{j \in J} Q[A_j] = \sum_{j \in J} c_j P[A_j]$ and $P[A] = \sum_{j \in J} P[A_j]$, then $P[A] = 0$ implies that $Q[A] = 0$, so that $Q$ is absolutely continuous with respect to $P$ on $\mathcal{F}$.

Now suppose that $Q$ is absolutely continuous with respect to $P$ on $\mathcal{F}$. For $i \in \{1, \ldots, n\}$ if $P[A_i] = 0$ define $c_i := 0$ and otherwise $c_i := \frac{Q[A_i]}{P[A_i]}$. Then $\xi := \sum c_i I_{A_i}$ is clearly $\geq 0$, and the construction of $\xi$ implies that $E^P[\xi] = \sum c_i P[A_i] = \sum Q[A_i] = 1$ and $Q[A_i] = \int_{A_i} \xi dP$.

(b) Consider $\mathcal{F} = \sigma(A_1, A_2)$, with $A_1 \cup A_2 = \Omega$ and $A_1 \bigcap A_2 = \emptyset$. Consider the probability measures $Q$ and $P$ defined by $Q[A_1] = 1, Q[A_2] = 0$ and $P[A_1] = P[A_2] = 1/2$. Then it is clear that $Q$ is absolutely continuous with respect to $P$, but $P$ and $Q$ are not equivalent.

**Exercise 6.2**

(a) Suppose that $(\xi_k)_{k\in\mathbb{N}}$ are independent integrable random variables with expectation 1. Define the process $X = \{X_n\}_{n\in\mathbb{N}_0}$ by $X_n := \prod_{k=1}^{n} \xi_k$. Show that $X$ is a martingale for its natural filtration.

(b) Give an example of a stochastic process in discrete time which is not locally bounded.

**Solution 6.2**

(a) Denote the natural filtration of $X$ as $\mathbb{F} = (\mathcal{F}_k)_{k\in\mathbb{N}_0}$. Then X is adapted by definition, and integrable because the product of independent integrable random variables is integrable. Moreover, by the definition of $X$ and the properties of conditional expectation, we have

$$E[X_n - X_{n-1}|\mathcal{F}_{n-1}] = \Big(\prod_{k=1}^{n-1} \xi_k\Big)E[\xi_n - 1|\mathcal{F}_{n-1}] = \Big(\prod_{k=1}^{n-1} \xi_k\Big)E[\xi_n - 1] = 0.$$

Thus $X$ is a martingale.

(b) Start with a sequence $(\xi_k)_{k\in\mathbb{N}}$ of nonnegative random variables and define the process $X = (X_n)_{n\in\mathbb{N}_0}$ by $X_n = \sum_{k=1}^{n} \xi_k$. Then $X_0 = 0$, so $X^\tau = X^\tau I_{\{\tau>0\}}$, and so it is enough to consider any stopping times $\tau \geq 1$. Then $X_\tau \geq X_1 I_{\{\tau\geq 1\}} = \xi_1$ because the $\xi_k$ are nonnegative. So if $\xi_1$ is unbounded, then $X^\tau$ cannot be bounded for any stopping time $\tau \geq 1$, and so $X$ is not locally bounded.

**Exercise 6.3**

Consider a sequence $(\xi_k)_{k\in\mathbb{N}}$ of i.i.d. random variables with $\xi_1 \sim \mathcal{N}(0,1)$. Define the process $M = (M_n)_{n\in\mathbb{N}_0}$ by $M_n := \sum_{k=1}^n \xi_k$. Let $\mathbb{F} = (\mathcal{F}_k)_{k\in\mathbb{N}_0}$ be the natural filtration of $M$.

(a) Show that $X_n := M_n^2 - n, n \in \mathbb{N}_0$, is a martingale.

(b) Show that $Y_n := \exp(M_n - n/2), n \in \mathbb{N}_0$, is a martingale.

(c) For any bounded predictable process $\alpha = (\alpha_i)_{i\in\mathbb{N}}$ define $N := \alpha \cdot M$ so that $N_k = \sum_{i=1}^{k} \alpha_i(M_i - M_{i-1})$ for $k \in \mathbb{N}_0$. Define also $\langle N\rangle = (\langle N\rangle_k)_{k\in\mathbb{N}_0}$ by $\langle N\rangle_k := \sum_{i=1}^{k} \alpha_i^2$. Show that $X := N^2 - \langle N\rangle$ and $Y := \exp(N - \langle N\rangle/2)$ are martingales.

**Solution 6.3**

Questions (a) and (b) follow directly from (c) by taking $\alpha \equiv 1$. So we only prove (c). Clearly, $N$ is adapted and $\langle N\rangle$ is predictable for $\mathbb{F}$. Moreover, both $X$ and $Y$ are integrable because $\alpha$ is bounded and the $\xi_i$ have all exponential moments. For the martingale property, note that $N$ is a martingale, so that

$$E[\Delta(N^2)_k|\mathcal{F}_{k-1}] = E[(\Delta N_k)^2|\mathcal{F}_{k-1}] =$$

$$= \alpha_k^2 E[(\Delta M_k)^2|\mathcal{F}_{k-1}] = \alpha_k^2 E[\xi_k^2] = \alpha_k^2 = \Delta\langle N\rangle_k.$$

This shows that $X = N^2 - \langle N\rangle$ is a martingale.

Similarly, using that $\alpha_k$ is $\mathcal{F}_{k-1}$ - measurable, $\xi_k$ is independent of $\mathcal{F}_{k-1}$ and $\Delta N_k = \alpha_k \Delta M_k = \alpha_k \xi_k$, we get $E[Y_k/Y_{k-1}|\mathcal{F}_{k-1}] = E[\exp(\Delta N_k - \Delta\langle N\rangle_k/2)|\mathcal{F}_{k-1}] = E[\exp(\alpha_k\xi_k - \alpha_k^2/2)|\mathcal{F}_{k-1}] = E[\exp(\lambda\xi_k - \lambda^2/2)]|_{\lambda=\alpha_k} = 1$ because $\xi_k \sim \mathcal{N}(0,1)$. So $Y$ is also a martingale.

**Exercise 6.4**

Using the notions from the lecture, show that the following are equivalent:

(a) $S = S^0(1, X)$ satisfies NA.

(b) $\mathcal{G}_{adm} \bigcap L^0_+ = \{0\}$.

(c) $\mathcal{C}_{adm} \bigcap L^0_+ = \{0\}$.

**Solution 6.4** By Exercise 4.3(c), NA for $S$ and NA for $S/S^0$ are equivalent. By Proposition II.2.1, NA for $S/S^0$ is equivalent to $\mathcal{G} \bigcap L^0_+(\mathcal{F}_T) = \{0\}$, and this is also equivalent to $\mathcal{G} \bigcap L^0_+ = \{0\}$ because $\mathcal{G} \subseteq L^0(\mathcal{F}_T)$. Because $0 \in \mathcal{G}_{adm} \subseteq \mathcal{G}$, the condition $\mathcal{G} \bigcap L^0_+ = \{0\}$ implies $\mathcal{G}_{adm} \bigcap L^0_+ = \{0\}$, and so we get "$(a) \Rightarrow (b)$".
Conversely, if we look at the proof of "5) $\Rightarrow$ 1)" for Proposition II.2.1, we can see that a slight modification also proves that $\mathcal{G}_{adm} \bigcap L^0_+ = \{0\}$ implies NA for $S/S^0$. Indeed, if $\psi$ is $a$-admissible in that argument, then

$$V(\bar{\psi}) = V(\psi) - V_0(\psi)V(\psi^*) = V(\psi) - V_0(\psi) \geq V(\psi)$$

shows that $\bar{\psi}$ is also $a$-admissible, and the rest of the argument goes as before. So we also have "$(b) \Rightarrow (a)$".
Because $0 \in \mathcal{G}_{adm} \subseteq \mathcal{C}_{adm}$, we clearly have "$(c) \Rightarrow (b)$". Conversely, if $c \in \mathcal{C}_{adm}$, then $c = g - Y$ with $g \in \mathcal{G}_{adm}$ and $Y \geq 0$. If also $c \in L^0_+$, then $c \geq 0$ and $g = c + Y \geq 0$ so that $g \in \mathcal{G}_{adm} \bigcap L^0_+$. By (b), we then have $0 = g = c + Y$ with $c \geq 0, Y \geq 0$, and therefore also $c = 0$. This shows that $\mathcal{C}_{adm} \bigcap L^0_+ = \{0\}$ and hence "$(b) \Rightarrow (c)$".