Introduction to Mathematical Finance

Exercise sheet 7

Exercise 7.1

(a) Let U be a standard normal random variable $U \sim \mathcal{N}(0,1)$. Consider a market with $T = 1, X_0 = 1$ and $X_1 = e^{\sigma U + \mu}$ for some constants $\mu, \sigma \in \mathbb{R}$, $\sigma \neq 0$. Construct an EMM for X.

(b) Consider a market with $X_0 = 1$ and $X_k := \prod_{j=1}^{k} e^{R_j}$, $k = 1, \ldots, T$, where $R_1, ..., R_T$ are i.i.d. with $R_1 \sim \mathcal{N}(\mu, \sigma^2)$ for some constants $\mu, \sigma \in \mathbb{R}$, $\sigma \neq 0$. Let $\mathbb{F} = (\mathcal{F}_k)_{k=1}^{T}$ be the natural filtration of X. Show that the market is arbitrage-free.

Exercise 7.2

Consider a market $(1, X)$ with $X_0 = 1$ and $X_k = \prod_{j=1}^{k} R_j$ for $k = 1, ..., T$, where $R_1, ..., R_T$ are i.i.d. under P and > 0. The filtration $\mathbb{F}$ is generated by X. Suppose that we have an EMM Q for X of the form

$$\frac{dQ}{dP} = \prod_{k=1}^{T} g_1(R_k)$$

for a measurable function $g_1 : (0, \infty) \mapsto (0, \infty)$. Show that $R_1, ..., R_T$ are also i.i.d. under Q.

Exercise 7.3 Consider an undiscounted financial market in finite discrete time with two assets S^0, S^1 which are both strictly positive. Suppose that the market is arbitrage-free and denote by $\mathbb{P}(S^i)$ for $i = 0, 1$ the set of all equivalent martingale measures for S^i-discounted prices.

(a) Take any $Q \in \mathbb{P}(S^0)$ and define R by $\frac{R}{Q} := \frac{S_T^1}{S_T^0} / \frac{S_0^1}{S_0^0}$. Prove that $R \in \mathbb{P}(S^1)$.

(b) Take any $Q =: Q^{S^0} \in \mathbb{P}(S^0)$ and define $Q^{S^1} := R$ as in (a). For any $H \in L_+^0(\mathcal{F}_T)$, prove the *change of numéraire* formula

$$S_k^0 E_{Q^{S^0}} \left[\frac{H}{S_T^0} \middle| \mathcal{F}_k \right] = S_k^1 E_{Q^{S^1}} \left[\frac{H}{S_T^1} \middle| \mathcal{F}_k \right]$$

for $k = 0, 1, ..., T$.