Introduction to Mathematical Finance

Exercise sheet 7

Exercise 7.1

(a) Let U be a standard normal random variable $U \sim \mathcal{N}(0,1)$. Consider a market with $T = 1, X_0 = 1$ and $X_1 = e^{\sigma U + \mu}$ for some constants $\mu, \sigma \in \mathbb{R}$, $\sigma \neq 0$. Construct an EMM for X.

(b) Consider a market with $X_0 = 1$ and $X_k := \prod_{j=1}^{k} e^{R_j}$, $k = 1, \dots, T$, where $R_1, ..., R_T$ are i.i.d. with $R_1 \sim \mathcal{N}(\mu, \sigma^2)$ for some constants $\mu, \sigma \in \mathbb{R}$, $\sigma \neq 0$. Let $\mathbb{F} = (\mathcal{F}_k)_{k=1}^T$ be the natural filtration of X. Show that the market is arbitrage-free.

Solution 7.1

(a) It is enough to construct a positive random variable D with $E[D] = 1$ and $E[X_1 D] = 1$, because then we can define the EMM by $\frac{dQ}{dP} := D$. Consider D of the form $D = \exp(\alpha U + \beta)$ with some constants $\alpha, \beta \in \mathbb{R}$. Then we have $E[D] = \exp(\beta + \alpha^2/2)$; so $E[D] = 1$ if and only if $\beta = -\alpha^2/2$. From $E[X_1 D] = \exp(\mu + \beta + (\sigma + \alpha)^2/2)$, we conclude that $E[X_1 D] = 1$ if and only if $\mu + \beta + (\sigma + \alpha)^2/2 = 0$. Thus we have both $E[D] = 1$ and $E[X_1 D] = 1$ if and only if

$$\begin{cases} \beta = -\frac{\alpha^2}{2} \\ \mu + \beta + (\sigma + \alpha)^2/2 = 0 \end{cases},$$

which gives us, after substituting the first equation into the second, that

$$\begin{cases} \beta = -\frac{\alpha^2}{2} \\ \alpha = -\frac{\mu + \sigma^2/2}{\sigma} \end{cases}, \tag{1}$$

and $\frac{dQ}{dP} := D$ with these parameters defines the EMM for X.

(b) For $k = 1, \dots, T$, we have $X_k = \prod_{j=1}^{k} e^{R_j} = \prod_{j=1}^{k} e^{\sigma U_j + \mu}$, where $U_j := (R_j - \mu)/\sigma$ are i.i.d. with $U_1 \sim \mathcal{N}(0,1)$. Set $D_k := \exp(\alpha U_k + \beta)$, where α, β are defined in (a) by (1). Then with the same arguments as in (1) we have that $E[D_k | \mathcal{F}_{k-1}] = 1$ and $E[D_k \frac{X_k}{X_{k-1}} | \mathcal{F}_{k-1}] = E[D_k e^{R_k} | \mathcal{F}_{k-1}] = 1$. Hence

$$\frac{dQ}{dP} := \prod_{k=1}^{T} D_k$$

yields an EMM for X, and so the market is arbitrage-free.

Exercise 7.2

Consider a market $(1, X)$ with $X_0 = 1$ and $X_k = \prod_{j=1}^{k} R_j$ for $k = 1, ..., T$, where $R_1, ..., R_T$ are i.i.d. under P and > 0. The filtration $\mathbb{F}$ is generated by X. Suppose that we have an EMM Q for X of the form

$$\frac{dQ}{dP} = \prod_{k=1}^{T} g_1(R_k)$$

for a measurable function $g_1 : (0, \infty) \mapsto (0, \infty)$. Show that $R_1, ..., R_T$ are also i.i.d. under Q.

Solution 7.2 Random variables $Z_1, ..., Z_N$ are independent under P if and only if for any measurable and bounded functions $f_1, ..., f_N$, we have

$$E^P\Big[\prod_{j=1}^{N} f_j(Z_j)\Big] = \prod_{j=1}^{N} E^P\Big[f_j(Z_j)\Big].$$

Consider any measurable and bounded functions $f_1, ..., f_T$. Then

$$E^Q\left[\prod_{j=1}^{T} f_j(R_j)\right] = E^P\left[\prod_{j=1}^{T} f_j(R_j) \prod_{j=1}^{T} g_1(R_j)\right] = E^P\left[\prod_{j=1}^{T} f_j(R_j) g_1(R_j)\right].$$

Since $R_1, ..., R_T$ are independent under P, then for any measurable functions $z_1, ..., z_T$ such that $E^P[z_j(R_j)] < \infty$ for $j = 1, ..., T$, we have

$$E^P\left[\prod_{j=1}^{T} z_j(R_j)\right] = \prod_{j=1}^{T} E^P\left[z_j(R_j)\right].$$

Hence, by taking $z_j := f_j \cdot g_1$, we derive that

$$E^P\left[\prod_{j=1}^{T} f_j(R_j) g_1(R_j)\right] = \prod_{j=1}^{T} E^P\left[f_j(R_j) g_1(R_j)\right],$$

and so

$$E^Q\left[\prod_{j=1}^{T} f_j(R_j)\right] = \prod_{j=1}^{T} E^P\left[f_j(R_j) g_1(R_j)\right].$$

On the other hand , since $R_1, ..., R_T$ are i.i.d. under P, we have

$$1 = E^P\left[\frac{dQ}{dP}\right] = E^P\left[\prod_{j=1}^{T} g_1(R_j)\right] = \prod_{j=1}^{T} E^P\left[g_1(R_j)\right] = \Big(E^P\left[g_1(R_1)\right]\Big)^T,$$

hence $E^P\left[g_1(R_j)\right] = E^P\left[g_1(R_1)\right] = 1$. Finally,

$$\prod_{j=1}^{T} E^Q\left[f_j(R_j)\right] = \prod_{j=1}^{T} E^P\left[f_j(R_j) \prod_{j=1}^{T} g_1(R_j)\right] =$$

$$= \prod_{j=1}^{T} E^P \left[f_j(R_j) g_1(R_j) \right] \prod_{i \neq j} E^P \left[g_1(R_i) \right] = \prod_{j=1}^{T} E^P \left[f_j(R_j) g_1(R_j) \right].$$

Thus we proved that

$$E^Q \left[\prod_{j=1}^{T} f_j(R_j) \right] = \prod_{j=1}^{T} E^P \left[f_j(R_j) g_1(R_j) \right] = \prod_{j=1}^{T} E^Q \left[f_j(R_j) \right],$$

hence $R_1, ..., R_T$ are independent under Q.

Since $R_1, ..., R_T$ are i.i.d. under P, then for $j = 1, ..., T$ and any measurable and bounded function f, we have

$$E^Q[f(R_j)] = E^P[f(R_j) g_1(R_j)] \prod_{i \neq j} E^P \left[g_1(R_i) \right] = E^P[f(R_j) g_1(R_j)] =$$

$$= E^P[f(R_1) g_1(R_1)],$$

hence $R_1, ..., R_T$ are identically distributed under Q.

Above we used the property that random variables $\xi_1, ..., \xi_n$ are identically distributed under Q if and only if for any measurable and bounded function f we have $E^Q[f(\xi_j)] = E^Q[f(\xi_1)]$ for $j = 1, ..., n$. To prove this property it is enough to apply the equality $E^Q[f(\xi_j)] = E^Q[f(\xi_1)]$ to the functions $f(x) = f_y(x) := I_{x \leq y}$.

Exercise 7.3 Consider an undiscounted financial market in finite discrete time with two assets S^0, S^1 which are both strictly positive. Suppose that the market is arbitrage-free and denote by $\mathbb{P}(S^i)$ for $i = 0, 1$ the set of all equivalent martingale measures for S^i-discounted prices.

(a) Take any $Q \in \mathbb{P}(S^0)$ and define R by $\frac{R}{Q} := \frac{S_T^1}{S_T^0} / \frac{S_0^1}{S_0^0}$. Prove that $R \in \mathbb{P}(S^1)$.

(b) Take any $Q =: Q^{S^0} \in \mathbb{P}(S^0)$ and define $Q^{S^1} := R$ as in (a). For any $H \in L_+^0(\mathcal{F}_T)$, prove the *change of numéraire* formula

$$S_k^0 E_{Q^{S^0}}\left[\frac{H}{S_T^0}\middle|\mathcal{F}_k\right] = S_k^1 E_{Q^{S^1}}\left[\frac{H}{S_T^1}\middle|\mathcal{F}_k\right]$$

for $k = 0, 1, ..., T$.

Solution 7.3

(a) R defined by the Radon–Nikodým derivative $\frac{R}{Q} := \frac{S_T^1}{S_T^0} / \frac{S_0^1}{S_0^0}$ defines a measure equivalent to Q and therefore equivalent to P. Indeed

$$\frac{R}{Q} := \frac{S_T^1 S_0^0}{S_T^0 S_0^1} > 0.$$

Moreover R is a probability measure since

$$\mathbb{E}_R\left[\mathbb{1}_\Omega\right] = \mathbb{E}_Q\left[\frac{R}{Q}\mathbb{1}_\Omega\right] = \mathbb{E}_Q\left[\frac{S_T^1 S_0^0}{S_T^0 S_0^1}\right] = \mathbb{E}_Q\left[\mathbb{E}_Q\left[\frac{S_T^1}{S_T^0} \mid \mathcal{F}_0\right]\frac{S_0^0}{S_0^1}\right] = 1$$

Finally the S^1 discounted prices are martingales under R:

$$\mathbb{E}_R\left[\frac{S_T^0}{S_T^1} \mid \mathcal{F}_k\right] = \frac{S_k^0}{S_k^1}$$

where the last equality uses the change of numéraire formula from b).

(b) We use Bayes rule to compute

$$\begin{aligned}
S_k^1 E_{Q^{S^1}}\left[\frac{H}{S_T^1}\middle|\mathcal{F}_k\right] &= S_k^1 \frac{E_{Q^{S^0}}[H\frac{dQ^{S^1}}{dQ^{S^0}}/S_T^1|\mathcal{F}_k]}{E_{Q^{S^0}}\left[\frac{dQ^{S^1}}{dQ^{S^0}}\middle|\mathcal{F}_k\right]} \\
&= S_k^1 \frac{E_{Q^{S^0}}[H\frac{dQ^{S^1}}{dQ^{S^0}}/S_T^1|\mathcal{F}_k]}{\frac{S_k^1}{S_k^0}/\frac{S_0^1}{S_0^0}} \\
&= S_k^0 \frac{E_{Q^{S^0}}\left[H\frac{S_T^1}{S_T^0}\frac{S_0^0}{S_0^1}\frac{1}{S_T^1}|\mathcal{F}_k\right]}{1/\frac{S_0^1}{S_0^0}} \\
&= S_k^0 E_{Q^{S^0}}\left[\frac{H}{S_T^0}\middle|\mathcal{F}_k\right].
\end{aligned}$$