Introduction to Mathematical Finance

Exercise sheet 8

Exercise 8.1 Consider the one-step market with 1 risky asset S^1 and 1 riskless asset S^0, which prices are given by

$$S_0^0 = 1, \qquad S_1^0 = 1 + r,$$

$$S_0^1 = 100, \qquad S_1^1 = 100(1 + \Delta X),$$

where $r > 0$ is a constant and $\Delta X \sim \mathcal{N}(\mu, \sigma^2)$. Consider the utility function

$$U(x) = \frac{1 - e^{-ax}}{a}, \quad a > 0.$$

Suppose that at time $t = 0$, we are given the amount of money A to invest in this market. Find an optimal strategy $(A - \pi, \pi)$ which allocates the amountπ to the risky asset and $A - \pi$ to the riskless asset and maximizes the expected utility of the portfolio wealth.

Exercise 8.2

(a) For a twice differentiable utility function $U : (0, \infty) \to \mathbb{R}$, the so-called *absolute risk aversion* is given by

$$A(x) = -\frac{U''(x)}{U'(x)}.$$

Characterize all utility functions $U = U^a$ with constant absolute risk aversion $A(x) \equiv a > 0$. Normalize the functions so that $U^a(0) = 0$ and $(U^a)'(0) = 1$.

(b) Let $(\Omega, \mathcal{F}, P)$ be a general probability space. Assume the standard model on $(\Omega, \mathcal{F}, P)$. Suppose that U is strictly increasing. Show that if there is an arbitrage opportunity, then there is no solution to the utility maximisation problem

$$\max_{\vartheta \in \Theta} E\left[U(x + G_T(\vartheta))\right].$$

Exercise 8.3 For a twice differentiable utility function $U : (0, \infty) \to \mathbb{R}$, the so-called *relative risk aversion* is given by

$$R(x) = -\frac{xU''(x)}{U'(x)}.$$

(a) Characterize all utility functions $U = U^\gamma$ with constant relative risk aversion $R(x) \equiv \gamma$. Normalize the functions so that $U^\gamma(1) = 0$ and $(U^\gamma)'(1) = 1$.

(b) Verify that $\lim_{\gamma \to 1} U^\gamma(x) = U^1(x)$ for all x.

Exercise 8.4

(a) Consider a market without arbitrage. Show that for every countable family of contingent claims $H_n \in L^0_+(\Omega, \mathcal{F}_T, P)$, $\forall n \in \mathbb{N}$ there exists an equivalent martingale measure Q such that $H_n \in L^1(\Omega, \mathcal{F}_T, Q)$ for all n.

(b) Construct an example for a family of uniformly bounded random variables whose pointwise supremum is not a random variable.