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Introduction to Mathematical Finance

Exercise sheet 8

Exercise 8.1 Consider the one-step market with 1 risky asset S^1 and 1 riskless asset S^0, which prices are given by

$$S_0^0 = 1, \qquad S_1^0 = 1 + r,$$
$$S_0^1 = 100, \qquad S_1^1 = 100(1 + \Delta X),$$

where $r > 0$ is a constant and $\Delta X \sim \mathcal{N}(\mu, \sigma^2)$. Consider the utility function

$$U(x) = \frac{1 - e^{-ax}}{a}, \quad a > 0.$$

Suppose that at time $t = 0$, we are given the amount of money A to invest in this market. Find an optimal strategy $(A - \pi, \pi)$ which allocates the amountπ to the risky asset and $A - \pi$ to the riskless asset and maximizes the expected utility of the portfolio wealth.

Solution 8.1

Denote as ϑ the amount of money invested in the risky asset. Hence the amount of money invested in the riskless asset is $A - \pi$ and the portfolio wealth associated with this strategy is

$$V(\pi) = A + \pi \Delta X + (A - \pi) r \sim \mathcal{N}(\pi(\mu - r) + A(1 + r), \pi^2 \sigma^2).$$

The expected utility of the portfolio is

$$E\Big[U\Big(V(\pi)\Big)\Big] = \frac{1}{a} - \frac{1}{a} E\Big[e^{-aV(\pi)}\Big];$$

hence in order to maximize the expected utility of the portfolio wealth, we need to find a strategy which minimizes the value $E\Big[e^{-aV(\pi)}\Big]$. Since

$$-aV(\pi) \sim \mathcal{N}\Big(- a(\pi(\mu - r) + A(1 + r)), a^2 \pi^2 \sigma^2 \Big),$$

we get

$$E\Big[e^{-aV(\pi)}\Big] = -a(\pi(\mu - r) + A(1 + r)) + \frac{a^2 \pi^2 \sigma^2}{2},$$

which reaches its minimum at the point

$$\pi^* = \frac{\mu - r}{a\sigma^2}.$$

Thus, the optimal strategy is $(A - \pi^*, \pi^*)$.

Exercise 8.2

(a) For a twice differentiable utility function $U : (0, \infty) \to \mathbb{R}$, the so-called *absolute risk aversion* is given by

$$A(x) = -\frac{U''(x)}{U'(x)}.$$

Characterize all utility functions $U = U^a$ with constant absolute risk aversion $A(x) \equiv a > 0$. Normalize the functions so that $U^a(0) = 0$ and $(U^a)'(0) = 1$.

(b) Let $(\Omega, \mathcal{F}, P)$ be a general probability space. Assume the standard model on $(\Omega, \mathcal{F}, P)$. Suppose that U is strictly increasing. Show that if there is an arbitrage opportunity, then there is no solution to the utility maximisation problem

$$\max_{\vartheta \in \Theta} E\left[U(x + G_T(\vartheta))\right].$$

Solution 8.2

(a) Fix $a > 0$ and write $U := U^a$.

From the ODE

$$-\frac{U''(x)}{U'(x)} = a,$$

we get that

$$U(x) = C_1 e^{-ax} + C_2.$$

From $U'(0) = 1$, it follows that $C_1 = -1/a$, and from $U(0) = 0$, we get that $C_1 + C_2 = 0$, hence $C_2 = 1/a$. Thus, the normalized utility function with constant absolute risk aversion $a > 0$ is given by

$$U(x) = \frac{1 - e^{-ax}}{a}.$$

(b) Suppose that ϑ^* is an optimiser and ϑ^A is an arbitrage opportunity. Then

$$x + G_T(\vartheta^*) \leq x + G_T(\vartheta^* + \vartheta^A) \quad P\text{-a.s. with } P\left[G_T(\vartheta^*) < G_T(\vartheta^* + \vartheta^A)\right] > 0.$$

Set $\vartheta' := \vartheta^* + \vartheta^A$. Because U is increasing, $U(x + G_T(\vartheta')) \geq U(x + G_T(\vartheta^*))$ P-a.s., and because U is strictly increasing, also $P[x+G_T(\vartheta') > x+G_T(\vartheta^*)] > 0$. So $E[U(x + G_T(\vartheta'))] > E[U(x + G_T(\vartheta^*))]$ which contradicts the optimality of ϑ^*.

Exercise 8.3 For a twice differentiable utility function $U : (0, \infty) \to \mathbb{R}$, the so-called *relative risk aversion* is given by

$$R(x) = -\frac{xU''(x)}{U'(x)}.$$

(a) Characterize all utility functions $U = U^\gamma$ with constant relative risk aversion $R(x) \equiv \gamma$. Normalize the functions so that $U^\gamma(1) = 0$ and $(U^\gamma)'(1) = 1$.

(b) Verify that $\lim_{\gamma\to 1} U^\gamma(x) = U^1(x)$ for all x.

Solution 8.3 Fix γ and write $U := U^\gamma$.

(a) Since the ODE

$$\frac{U''(x)}{U'(x)} = -\frac{\gamma}{x}$$

is separable, we find U' from

$$\ln U' = \int \frac{1}{U'} dU' = -\int \frac{\gamma}{y} dy = -\gamma \ln x + C.$$

From $U'(1) = 1$, we obtain $C = 0$. Hence,

$$U'(x) = x^{-\gamma}.$$

Integrate again to get

$$U(x) = \begin{cases} \frac{x^{1-\gamma}}{1-\gamma} + D, & \text{if } \gamma \neq 1, \\ \ln x + D, & \text{if } \gamma = 1. \end{cases}$$

The condition $U(1) = 0$ gives $D = -1/(1-\gamma)$ and $D = 0$ for the two cases, respectively. The utility function $U = U^\gamma$ is therefore given by

$$U^\gamma(x) = \begin{cases} \frac{x^{1-\gamma}-1}{1-\gamma}, & \text{if } \gamma \neq 1, \\ \ln x, & \text{if } \gamma = 1. \end{cases}$$

If we also want U to be concave, we have to impose $\gamma \geq 0$.

(b) We employ L'Hôpital's rule to get

$$\lim_{\gamma\to 1} \frac{x^{1-\gamma}-1}{1-\gamma} = \lim_{\gamma\to 1} \frac{\frac{d}{d\gamma}\gamma(x^{1-\gamma}-1)}{\frac{d}{d\gamma}\gamma(1-\gamma)} = \lim_{\gamma\to 1} \frac{-x^{1-\gamma}\ln x}{-1} = \ln x,$$

which is what we wanted to show.

Exercise 8.4

(a) Consider a market without arbitrage. Show that for every countable family of contingent claims $H_n \in L^0_+(\Omega, \mathcal{F}_T, P)$, $\forall n \in \mathbb{N}$ there exists an equivalent martingale measure Q such that $H_n \in L^1(\Omega, \mathcal{F}_T, Q)$ for all n.

(b) Construct an example for a family of uniformly bounded random variables whose pointwise supremum is not a random variable.

Solution 8.4

(a) First we show how can we find such a martingale measure for one contingent claim H.
Note that since $H \geq 0$,

$$E_P\left[\frac{1}{1+|H|}\right] \in (0,1].$$

Hence, P' defined by

$$\frac{dP'}{dP} = \frac{1}{1+|H|}\Big/ E_P\left[\frac{1}{1+|H|}\right]$$

is well-defined. Furthermore,

$$E_{P'}\left[|H|\right] = E_P\left[\frac{|H|}{1+|H|}\right]\Big/ E_P\left[\frac{1}{1+|H|}\right] \leq 1\Big/ E_P\left[\frac{1}{1+|H|}\right] < \infty,$$

showing that $H \in L^1(\Omega, \mathcal{F}_T, P')$. Since $P \approx P'$, the market is free of arbitrage also under P'. Thus by Theorem II.3.1, there exists an EMM Q with $\frac{dQ}{dP'} \in L^\infty$. Therefore, H is also Q-integrable.

Note that we can use the same construction as above to find an EMM Q such that any *finite* set of contingent claims $(H_k)_{k=1}^N$ is Q-integrable. Indeed, introduce $H := \sum_{k=1}^N |H_k|$. Then, by repeating the reasoning above, we will get an EMM Q such that all H_k are Q-integrable.

To get $P' \approx P$ which makes all the $(H_k)_{k=1}^\infty$ simultaneously integrable, we start with H of the form

$$H = \sum_{n=1}^\infty c_n H_n, \qquad \text{with constants } c_n > 0$$

and then use

$$\frac{dP'}{dP} := C\frac{1}{1+|H|}.$$

Provided that $|H| < \infty$ P-a.s., this gives $Q \approx P' \approx P$ with Q an EMM for X and $H \in L^1(\Omega, \mathcal{F}_T, Q)$ for all $n \in \mathbb{N}$ because $|H_n| \leq 1/c_n|H|$.

To check that $|H| < \infty$ P-a.s. can be achieved by a suitable choice of c_n, start with constants $a_n > 0$, such that $\sum_{n=1}^{\infty} P[H_n > a_n] < \infty$. By Borel–Cantelli's Lemma, we then have with probability 1 that $H_n \leq a_n$ eventually, i.e.

$$H_n(\omega) \leq a_n \quad \text{for all } n \geq n_0(\omega)$$

for P-almost all ω. So if we take $c_n := 2^{-n}/a_n$, we get

$$\sum_{n=1}^{\infty} c_n H_n(\omega) \leq \sum_{n=1}^{n_0(\omega)} c_n H_n(\omega) + \sum_{n>n_0(\omega)} 2^{-n} < \infty$$

for P-almost all ω. (The idea of this construction goes back to Dellacherie (1978).)

(b) Let $\Omega = [0,1]$ with the Borel sigma-algebra and P the Lebesgue measure. Let $V \subseteq \Omega$ be any set which is not Lebesgue-measurable, for example the Vitali set, and $(X_v)_{v \in V}$ the family of random variables X_v defined by

$$X_v = I_{\{v\}}.$$

Note that every X_v is indeed a random variable because $\{v\}$ is closed, hence a Borel set. However, the pointwise supremum is

$$\sup_{v \in V} X_v = I_V,$$

which is not measurable by construction.