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Introduction to Mathematical Finance

Exercise sheet 9

Exercise 9.1 Recall that an investment and consumption pair $(\psi, \tilde{c})$ with initial endowment $\tilde{v}_0$ is self-financing if $\psi_1 \cdot S_0 + \tilde{c}_0 = \tilde{v}_0$ and

$$\triangle \psi_{k+1} \cdot S_k + \tilde{c}_k = 0$$

for $k = 1, \ldots, T-1$. Define the undiscounted wealth by $\tilde{W}_0 = \tilde{v}_0$ and $\tilde{W}_k := \psi_k \cdot S_k$ for $k = 1, \ldots, T$, $W = \tilde{W}/S^0$ and $c = \tilde{c}/S^0$.

(a) Show in detail that $(\psi, \tilde{c})$ is self-financing if and only if

$$W_k = v_0 + \sum_{j=1}^{k} (\vartheta_j \cdot \triangle X_j - c_{j-1}) \quad \text{for } k = 0, \ldots, T.$$

(b) Show that the pair $(\psi, \tilde{c})$ with initial wealth $\tilde{v}_0$ is self-financing if and only if

$$\tilde{W}_k = \tilde{v}_0 + \sum_{j=1}^{k} \left(\vartheta_j \cdot \triangle S_j - \tilde{c}_{j-1} \right) \quad \text{for } k = 0, ..., T.$$

Exercise 9.2 Recall that for each $k \in \{0, 1, ..., T\}$, suitable $\mathcal{F}_k$-measurable v_k and $(\vartheta', c') \in \mathcal{A}$, we define the *remaining conditional expected utility* to be

$$R_k(v_k, \vartheta', c') := E\left[\sum_{j=k}^{T} U_c(c'_j) + U_w\left(v_k + \sum_{j=k+1}^{T} (\vartheta'_j \cdot \triangle X_j - c'_{j-1}) - c'_T \right) \middle| \mathcal{F}_k \right].$$

Recall that

$$\mathcal{A}_k(\vartheta, c) := \{(\vartheta', c') \in \mathcal{A} : \vartheta'_j = \vartheta_j \text{ for } j \leq k, c'_j = c_j \text{ for } j \leq k-1\}.$$

Show that for fixed $(\vartheta, c) \in \mathcal{A}$, we have

$$\operatorname{ess\,sup}_{(\vartheta', c') \in \mathcal{A}_k(\vartheta, c)} R_k(W_k^{v_0, \vartheta, c}, \vartheta', c') = \operatorname{ess\,sup}_{(\vartheta', c') \in \mathcal{A}} R_k(W_k^{v_0, \vartheta, c}, \vartheta', c').$$

Exercise 9.3

(a) Let (ϑ, c) be a self-financing investment and consumption pair and

$$W_k = W_k^{v_0, \vartheta, c} = v_0 + \sum_{j=1}^{k} (\vartheta_j \cdot \triangle X_j - c_{j-1})$$

for $k = 0, 1, \ldots, T$ the corresponding discounted wealth process. Show that if $W \geq -a$ for some constant a, then W is a Q-supermartingale for any ELMM Q for X.

(b) Let $U : \mathbb{R} \to \mathbb{R}$ be concave and consider for fixed $Q \in \mathbb{P}_{\mathrm{loc}}$ the problem of maximising $E_Q[U(W_T^{v_0,\vartheta,c} - c_T)]$ over all self-financing investment and consumption pairs. Assuming that each $U(W_\cdot^{v_0,\vartheta,c})$ is Q-integrable and that $j_0 := \sup_{(\vartheta,c)} E_Q[U(W_T^{v_0,\vartheta,c} - c_T)] < \infty$, show that the solution is given by $\vartheta \equiv 0$, $c \equiv 0$.