Introduction to Mathematical Finance

Exercise sheet 9

Exercise 9.1 Recall that an investment and consumption pair $(\psi, \tilde{c})$ with initial endowment $\tilde{v}_0$ is self-financing if $\psi_1 \cdot S_0 + \tilde{c}_0 = \tilde{v}_0$ and

$$\triangle\psi_{k+1} \cdot S_k + \tilde{c}_k = 0$$

for $k = 1, \dots, T-1$. Define the undiscounted wealth by $\tilde{W}_0 = \tilde{v}_0$ and $\tilde{W}_k := \psi_k \cdot S_k$ for $k = 1, \dots, T$, $W = \tilde{W}/S^0$ and $c = \tilde{c}/S^0$.

(a) Show in detail that $(\psi, \tilde{c})$ is self-financing if and only if

$$W_k = v_0 + \sum_{j=1}^{k} (\vartheta_j \cdot \triangle X_j - c_{j-1}) \quad \text{for } k = 0, \dots, T.$$

(b) Show that the pair $(\psi, \tilde{c})$ with initial wealth $\tilde{v}_0$ is self-financing if and only if

$$\tilde{W}_k = \tilde{v}_0 + \sum_{j=1}^{k} \left(\vartheta_j \cdot \triangle S_j - \tilde{c}_{j-1}\right) \quad \text{for } k = 0, ..., T.$$

Solution 9.1

(a) First discount the self-financing condition to get a condition in X, namely

$$\triangle\psi^0_{k+1} + \triangle\vartheta_{k+1} \cdot X_k + c_k = 0,$$
$$\psi^0_1 + \vartheta_1 \cdot X_0 + c_0 = v_0.$$

Using this,

$$\begin{aligned}
\triangle W_k &= \frac{\tilde{W}_k}{S^0_k} - \frac{\tilde{W}_{k-1}}{S^0_{k-1}} = \psi_k \cdot (1, X_k) - \psi_{k-1} \cdot (1, X_{k-1}) \\
&= \psi^0_k - \psi^0_{k-1} + \vartheta_k \cdot X_k - \vartheta_{k-1} \cdot X_{k-1} - \triangle\psi^0_k - \triangle\vartheta_k \cdot X_{k-1} - c_{k-1} \\
&= \vartheta_k \cdot \triangle X_k - c_{k-1}
\end{aligned}$$

for $k \geq 2$. Furthermore,

$$\triangle W_1 = \frac{\tilde{W}_1}{S^0_1} - v_0 = \psi^0_1 + \vartheta_1 \cdot X_1 - \psi^0_1 - \vartheta_1 \cdot X_0 - c_0 = \vartheta_1 \cdot X_1 - c_0.$$

Summing both results yields

$$W_k = v_0 + \sum_{j=1}^{k} (\vartheta_j \cdot \triangle X_j - c_{j-1}), \quad \text{for } k = 0, \dots, T.$$

(b) We have

$$\triangle \tilde{W}_1 = \psi_1 \cdot S_1 - \tilde{v}_0 = \psi_1 \cdot S_1 - \psi_1 \cdot S_0 - \tilde{c}_0 = \psi_1 \cdot \triangle S_1 - \tilde{c}_0$$

as well as

$$\triangle \tilde{W}_k = \psi_k \cdot S_k - \psi_{k-1} \cdot S_{k-1} = \psi_k \cdot \triangle S_k + \triangle \psi_k \cdot S_{k-1} = \psi_k \cdot \triangle S_k - \tilde{c}_{k-1}$$

for all $k = 2, \ldots, T$ if and only if $(\psi, \tilde{c})$ is self-financing. So we can sum up the increments to obtain

$$\tilde{W}_k = \tilde{W}_0 + \sum_{j=1}^{k} \triangle \tilde{W}_k = \tilde{v}_0 + \sum_{j=1}^{k} (\psi_k \cdot \triangle S_k - \tilde{c}_{k-1}).$$

Note that starting with these sums also gives the first two series of equalities, showing the equivalence.

Exercise 9.2 Recall that for each $k \in \{0, 1, ..., T\}$, suitable $\mathcal{F}_k$-measurable v_k and $(\vartheta', c') \in \mathcal{A}$, we define the *remaining conditional expected utility* to be

$$R_k(v_k, \vartheta', c') := E\left[\sum_{j=k}^{T} U_c(c'_j) + U_w\left(v_k + \sum_{j=k+1}^{T} (\vartheta'_j \cdot \triangle X_j - c'_{j-1}) - c'_T\right)\middle|\mathcal{F}_k\right].$$

Recall that

$$\mathcal{A}_k(\vartheta, c) := \{(\vartheta', c') \in \mathcal{A} : \vartheta'_j = \vartheta_j \text{ for } j \leq k, c'_j = c_j \text{ for } j \leq k-1\}.$$

Show that for fixed $(\vartheta, c) \in \mathcal{A}$, we have

$$\operatorname{ess\,sup}_{(\vartheta', c') \in \mathcal{A}_k(\vartheta, c)} R_k(W_k^{v_0, \vartheta, c}, \vartheta', c') = \operatorname{ess\,sup}_{(\vartheta', c') \in \mathcal{A}} R_k(W_k^{v_0, \vartheta, c}, \vartheta', c').$$

Solution 9.2 Define

$$\mathcal{A}_k^{\text{post}}(\vartheta, c) := \{(\vartheta', c') \in \mathcal{A} : \vartheta'_j = \vartheta_j \text{ for } j > k, c'_j = c_j \text{ for } j > k-1\}.$$

Note that $\mathcal{A} = \bigcup_{(\vartheta', c') \in \mathcal{A}} \mathcal{A}_k^{\text{post}}(\vartheta', c')$. For $(\vartheta', c') \neq (\vartheta'', c'')$, the sets $\mathcal{A}_k^{\text{post}}(\vartheta', c')$ and $\mathcal{A}_k^{\text{post}}(\vartheta'', c'')$ are either the same or disjoint. So for each distinct $\mathcal{A}_k^{\text{post}}(\vartheta', c')$, there exists $(\vartheta_0, c_0) \in \mathcal{A}_k(\vartheta, c)$ such that $\mathcal{A}_k^{\text{post}}(\vartheta', c') = \mathcal{A}_k^{\text{post}}(\vartheta_0, c_0)$. Hence,

$$\mathcal{A} = \bigcup_{(\vartheta', c') \in \mathcal{A}} \mathcal{A}_k^{\text{post}}(\vartheta', c') = \bigcup_{(\vartheta', c') \in \mathcal{A}_k(\vartheta, c)} \mathcal{A}_k^{\text{post}}(\vartheta', c').$$

Since $R_k(v_k, \vartheta', c')$ has the same value on each $\mathcal{A}_k^{\text{post}}(\vartheta', c')$, we have

$$\begin{aligned}
\operatorname{ess\,sup}_{(\vartheta', c') \in \mathcal{A}} R_k(v_k, \vartheta', c') &= \operatorname{ess\,sup} \left\{ R_k(v_k, \vartheta'', c'') : (\vartheta'', c'') \in \bigcup_{(\vartheta', c') \in \mathcal{A}_k(\vartheta, c)} \mathcal{A}_k^{\text{post}}(\vartheta', c') \right\} \\
&= \operatorname{ess\,sup}_{(\vartheta', c') \in \mathcal{A}_k(\vartheta, c)} R_k(v_k, \vartheta', c').
\end{aligned}$$

This also holds for $v_k = W_k^{v_0, \vartheta, c}$.

Exercise 9.3

(a) Let (ϑ, c) be a self-financing investment and consumption pair and

$$W_k = W_k^{v_0,\vartheta,c} = v_0 + \sum_{j=1}^{k} (\vartheta_j \cdot \triangle X_j - c_{j-1})$$

for $k = 0, 1, \ldots, T$ the corresponding discounted wealth process. Show that if $W \geq -a$ for some constant a, then W is a Q-supermartingale for any ELMM Q for X.

(b) Let $U : \mathbb{R} \to \mathbb{R}$ be concave and consider for fixed $Q \in \mathbb{P}_{\text{loc}}$ the problem of maximising $E_Q[U(W_T^{v_0,\vartheta,c} - c_T)]$ over all self-financing investment and consumption pairs. Assuming that each $U(W_\cdot^{v_0,\vartheta,c})$ is Q-integrable and that $j_0 := \sup\limits_{(\vartheta,c)} E_Q[U(W_T^{v_0,\vartheta,c} - c_T)] < \infty$, show that the solution is given by $\vartheta \equiv 0$, $c \equiv 0$.

Solution 9.3

(a) This is just Lemma II.6.2. Indeed, without loss of generality assume $a \geq 0$. Write $C_k := \sum_{j=0}^{k-1} c_j$. First note that $W_k + C_k = v_0 + (\vartheta \cdot X)_k$. By Proposition C.4, $W + C$ is a Q-local martingale. Since $W \geq -a$ and $C \geq 0$, we have that $W + C \geq -a$ is thus a Q-supermartingale and Q-integrable. So $C = W + C - W \leq W + C + a$ is Q-integrable. Then $W = W + C - C$ is Q-integrable and then a Q-supermartingale like $W + C$ since C is increasing.

(b) First we observe as in the lecture that

$$J_T(0,0) = \operatorname{ess\,sup}_{c'_T} U(v_0 - c'_T)$$

and $E_Q[J_T(0,0)] \geq \sup_{c'_T} E_Q[U(v_0 - c'_T)] \geq E_Q[U(v_0)]$. On the other hand, using that $W^{v_0,\vartheta,c}$ is Q-supermartingale and U is concave with $U(W_\cdot^{v_0,\vartheta,c}) \in L^1(Q)$ yields that $U(W_\cdot^{v_0,\vartheta,c} - c_T)$ is a Q-supermartingale for any $(\vartheta, c) \in \mathcal{A}$. Thus $U(v_0) \geq \operatorname{ess\,sup}_{(\vartheta',c') \in \mathcal{A}} E_Q[U(W_T^{v_0,\vartheta',c'} - c'_T) | \mathcal{F}_0]$ and

$$E_Q[U(v_0)] \geq E_Q[J_0(0,0)] \geq E_Q[J_T(0,0)] \geq E_Q[U(v_0)].$$

This implies that $J(0,0)$ is a supermartingale with constant expectation j_0, thus a martingale. Therefore, by the Martingale Optimality Principle: $\vartheta \equiv 0$, $c \equiv 0$ is optimal.