Introduction to Mathematical Finance

Exercise sheet 1

Please submit your solutions online until Wednesday 22:00, 04/03/2025.

Exercise 1.1

(a) Construct a market with arbitrage of the first kind but with no arbitrage of the second kind.

(b) Construct a market with arbitrage of the second kind but with no arbitrage of the first kind.

(c) Prove Proposition I.3.1. That is suppose there exists an asset $\mathcal{D}^l$ with $\mathcal{D}^l \geq 0$ and $\mathcal{D}^l \not\equiv 0$. Show that under this assumption, the market is arbitrage-free iff there is no arbitrage of first kind.

Solution 1.1

(a) Consider a market consisting of a single asset with $\pi = 0$, $\mathcal{D} = (1,2)^{\mathrm{tr}}$. Set $\vartheta = 1$. Clearly, $\mathcal{D}\vartheta = (1,2)^{\mathrm{tr}} \geq 0$ and $\mathcal{D}\vartheta(\{\omega_i\}) > 0$ for both $i = 1, 2$. Thus ϑ is an arbitrage opportunity of the first kind. However, since $\pi = 0$, there exists no arbitrage of the second kind.

(b) Consider the situation where $\pi = 1$ and $\mathcal{D} = (0,0)$. Then $\vartheta < 0$ would be an arbitrage of the second kind. But since $\mathcal{D}$ vanishes, we have for any $\tilde{\vartheta} \in \mathbb{R}$ that $\mathcal{D}\tilde{\vartheta} = (0,0)^{\mathrm{tr}}$. So there exists no arbitrage of the first kind.

(c) Suppose first that there is an asset $D^\ell \geq 0$ and $D^\ell \not\equiv 0$ and $\pi^\ell > 0$. Let ϑ be an arbitrage opportunity of the second kind. Set $\alpha = -\vartheta \cdot \pi / \pi^\ell > 0$. We consider a new strategy $\hat{\vartheta} = \vartheta + \alpha e_\ell$ where e_ℓ is the vector with 1 in its ℓth component and 0 elsewhere. Then $\hat{\vartheta} \cdot \pi = \vartheta \cdot \pi + \alpha \cdot \pi^\ell = 0$ and $\mathcal{D}\hat{\vartheta} = \mathcal{D}\vartheta + \alpha\mathcal{D}^\ell \geq 0$. Since $\mathcal{D}\vartheta \geq 0$ and $\alpha\mathcal{D}^\ell \geq 0$ with $\alpha\mathcal{D}^\ell \not\equiv 0$, we have $\mathcal{D}\hat{\vartheta} \geq 0$ and $\mathcal{D}\hat{\vartheta} \not\equiv 0$. Hence, $\hat{\vartheta}$ is an arbitrage opportunity of the first kind. The other implication is true in general.

Exercise 1.2 Let $\mathcal{C} := \mathbb{R} \times \mathbb{R}^K$ be the consumption space with the payoff matrix $\mathcal{D}$ and let e^i, π be an endowment, and a price vector, respectively. Recall the budget set

$$B(e^i, \pi) := \{c \in \mathcal{C} : \exists \vartheta \in \mathbb{R}^N \text{ with } c_0 \le e^i_0 - \vartheta \cdot \pi \text{ and } c_T \le e^i_T + \mathcal{D}\vartheta\}.$$

(a) Show $c \in B(e^i, \pi) \iff c - e^i \in B(0, \pi) \Longleftarrow c - e^i$ is attainable with 0 initial wealth.

(b) Show by an example that the converse of the second implication is not true in general.

Solution 1.2

(a) By definition, $c \in B(e^i, \pi)$ iff there exists $\vartheta \in \mathbb{R}^N$ with $c_0 \le e^i_0 - \vartheta \cdot \pi$ and $c_T \le e^i_T + \mathcal{D}\vartheta$. That is, $c_0 - e^i_0 \le -\vartheta \cdot \pi$ and $c_T - e^i_T \le \mathcal{D}\vartheta$, which means $c - e \in B(0, \pi)$.

Now if $c - e^i$ is attainable with 0 initial wealth, then there exists $\hat{\vartheta} \in \mathbb{R}^N$ such that $c_0 - e^i_0 = -\pi \cdot \hat{\vartheta}$ and $c_T - e^i_T = \mathcal{D}\hat{\vartheta}$ which shows $c - e^i \in B(0, \pi)$.

(b) The idea is simply to find a nonattainable consumption which still lies in the budget set. To do this, we consider a matrix without full rank. Let

$$\pi := \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathcal{D} := \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

Clearly $\mathcal{D}(\mathbb{R}^2) = \{(a, 2a)^{\mathrm{tr}} : a \in \mathbb{R}\}$. Take for instance $\vartheta = (1, 0)^{\mathrm{tr}}$, $c_T = e^i_T + (1, 1.5)^{\mathrm{tr}}$, and $c_0 = e^i_0 - 1$. Then

$$c_0 - e^i_0 \le -(1, 0) \cdot (1, 1) = -1,$$

$$c_T - e^i_T \le \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Thus, $c - e^i \in B(0, \pi)$. But clearly $(1, 1.5)^{\mathrm{tr}} \notin \mathcal{D}(\mathbb{R}^2)$, which shows $c - e^i$ cannot be attainable with 0 initial wealth.

Exercise 1.3 Suppose $\mathcal{D}$ is complete. Show that $B(e,\pi) = \mathcal{C}$ for all e if and only if there exists arbitrage of the second kind.

Solution 1.3 Begin by choosing $c_0 > 0$, $c_T \equiv 0$, and $e \equiv 0$. Then $c \in B(e,\pi)$ is equivalent to the existence of $\vartheta \in \mathbb{R}^N$ such that

$$c_0 \leq -\vartheta \cdot \pi \quad \text{or equivalently} \quad \vartheta \cdot \pi \leq -c_0 < 0$$

and

$$0 \leq \mathcal{D}\vartheta,$$

from which we conclude that there exists an arbitrage opportunity of the second kind. This implication does not need completeness.

For the converse, pick any $c \in \mathcal{C}$ and let ϑ^r be a strategy with $c_T - e_T = \mathcal{D}\vartheta^r$. This exists by completeness of $\mathcal{D}$. If

$$c_0 \leq e_0 - \pi \cdot \vartheta^r,$$

then $c \in B(e,\pi)$ and we are done. Otherwise, let ϑ^a be an arbitrage opportunity of the second kind with $\vartheta \cdot \pi = a < 0$. Then the strategy

$$\vartheta = \vartheta^r + \underbrace{\frac{c_0 - e_0 + \pi \cdot \vartheta^r}{-a}}_{>0} \vartheta^a$$

satisfies

$$c_0 \leq e_0 - \vartheta \cdot \pi = c_0$$

and

$$\mathcal{D}\vartheta + e_T = \mathcal{D}\vartheta^r + \left(\frac{c_0 - e_0 + \pi \cdot \vartheta^r}{-a}\right) \mathcal{D}\vartheta^a + e_T \geq \mathcal{D}\vartheta^r + e_T = c_T,$$

showing that $c \in B(e,\pi)$.