Mathematics for New Technologies in Finance

Exercise sheet 2

Exercise 2.1 (Weierstrass theorem [1])

(a) Construct a sequence of polynomials converges pointwisely but not uniformly on $[0, 1]$.

(b) Construct a sequence of polynomials converges uniformly to $x \mapsto |x|$ on $[-1, 1]$. (Hint: Corollary 2.3. in [1])

(c) Prove that ReLU can be approximated uniformly by polynomials on $[-1, 1]$.

(d) Use the universal approximation theory of shallow neural networks on $[0, 1]$ to prove the Weierstrass theorem.

Exercise 2.2 (Networks on discrete path spaces)

(a) Describe the space of paths $\omega : \{1, \ldots, T\} \to \mathbb{R}^d$ as $\mathbb{R}^{dT}$.

(b) Describe a shallow neural network, which depends on value at time t and on path information up to time t. Formulate a universal approximation theorem in this setting.

Exercise 2.3 (Linear Operators) Let K be a compact subset of $\mathbb{R}^d$.

(a) Let μ be a finite Borel measure on K. Prove that

$$\mathcal{L}_\mu(f) := \int_K f(x)\mu(dx) \tag{1}$$

for $f \in C(K, \mathbb{R})$ is a bounded linear functional.

(b) Let $\mathcal{L}$ be a positive linear functional on $C(K, \mathbb{R})$, i.e. $\mathcal{L}(f) \geq 0$ for $f \geq 0$. Prove that $\mathcal{L}$ is continuous.

Exercise 2.4 (Point-separating families)

(a) Let K be a compact subset of $\mathbb{R}^d$. Prove that

$$\mathcal{F} := \left\{ \mathcal{C}(K, \mathbb{R}) \ni f \mapsto \sum_{i=1}^{n} \lambda_i f(x_i) \mid \lambda_i \in \mathbb{R}, n \in \mathbb{N}, x_i \in K, i = 1, 2, ..., n \right\} \tag{2}$$

is point separating and additive.

(b) Prove that

$$\mathcal{F} := \left\{ \mathcal{C}_0^1([0, 1], \mathbb{R}) \ni X \mapsto \sum_{i=1}^{n} \lambda_i \int t^i dX_t \in \mathbb{R} \colon \forall \lambda_i \in \mathbb{R}, n \in \mathbb{N} \right\}$$

is a point-separating vector space. $\mathcal{C}_0^1([0, 1], \mathbb{R})$ is the space of the $\mathcal{C}^1$ function f on $[0, 1]$ with $f(0) = 0$.

Exercise 2.5 (Controlled ODEs as features on the path space) We aim to demonstrate that controlled ODEs define (non-linear) features on a path space, which we shall fix to $\mathcal{C}^1([0, T], \mathbb{R}^d))$. See notebook 1 for details.

References

[1] Sameer Chavan. Problems and notes: uniform convergence and polynomial approximation.