Mathematics for New Technologies in Finance

Solution sheet 3

Exercise 3.1 (Hölder-continuous functions as a weighted space) Fix $p \in]0,1]$ and consider the Hölder space $\mathcal{C}^p = \mathcal{C}^p([0,1],\mathbb{R}^d)$ of continuous function which are p-Hölder continuous, equipped with the uniform norm $\|\cdot\|_\infty$. Denote by

$$\|\omega\|_p := |\omega(0)| + \sup_{t\neq s} \frac{|\omega(t)-\omega(s)|}{|t-s|^p}$$

the Hölder norm on this space. Prove that this is a weighted space, under the topology induced by the uniform norm, with weight function

$$\rho(\omega) := 1 + \|\omega\|_p, \quad \forall \omega \in \mathcal{C}^p.$$

Exercise 3.2 (Controlled ODEs) Consider the controlled ODE: $X_0 = x \in \mathbb{R}$

$$dX_t^\theta = V^\theta(t, X_t^\theta)dt, \quad t \in [0,T]. \tag{1}$$

(a) Let

$$a_t = \frac{\partial X_T^\theta}{\partial X_t^\theta}. \tag{2}$$

Prove that

$$\frac{d}{dt}a_t = -\frac{\partial V^\theta}{\partial x}(t, X_t^\theta) \cdot a_t, \quad a_T = 1, \tag{3}$$

and relate a_t with $J_{t,T}$ in the lecture notebook.

(b) Prove that

$$\frac{d}{dt}\left(\frac{\partial X_t^\theta}{\partial \theta}a_t\right) = a_t \frac{\partial V^\theta}{\partial \theta}(t, X_t^\theta), \tag{4}$$

and

$$\frac{\partial X_T^\theta}{\partial \theta} = -\int_T^0 \frac{\partial X_T^\theta}{\partial X_t^\theta} \cdot \frac{\partial V^\theta}{\partial \theta}(t, X_t^\theta)dt. \tag{5}$$

(c) Is every feedforward neural network a discretization of controlled ODE?

Exercise 3.3 (Solution of CODEs as features) This exercise exemplifies how, for any input curve u, the solution of the associated controlled ODE (CODE) is a (non-trivial) feature of u. Consider any control $u \in \mathcal{C}^1([0,1],\mathbb{R})$ and define for $t \in [0,1]$ the linear CODE

$$\mathrm{d}X_t = (\lambda X_t + u_t)\mathrm{d}t, \quad X_0 = x$$

where $\lambda \in \mathbb{R}$.

(a) Solve the system for a generic $u \in \mathcal{C}^1([0,1],\mathbb{R})$.

(b) Explicitly compute the solution for $u = \sin$.

Exercise 3.4 (Dependence of a non-linear ODE on the starting value) For $t \in [0,1]$, consider the ODE

$$\mathrm{d}X_t = \sin(X_t)\mathrm{d}t, \quad X_0 = x$$

Compute the value of $\partial_x X_t$ in two different ways, respectively

(a) by direct calculation;

(b) using the operator $J_{s,t}$ introduced in the lecture notebook.