Mathematics for New Technologies in Finance

Exercise sheet 4

Exercise 4.1 (Signatures) Through this exercise, we let $E = \mathbb{R}^d$, J an interval on $\mathbb{R}$, and denote $\mathbf{Sig}_J : \mathcal{C}_0^1(J, E) \to \mathbf{T}(E)$ the signature map such that for all $X \in \mathcal{C}_0^1(J, E)$ and we let $\mathbf{Sig}_J^{(M)}$ denote the truncated signature map up to order M: $\mathbf{Sig}_J^{(M)}(X) = (1, \mathbf{s}_1, \cdots, \mathbf{s}_M) \in \mathbf{T}^{(M)}(E)$. Let $X \in \mathcal{C}_0^1([0, s], E)$ and $Y \in \mathcal{C}_0^1([s, t], E)$.

(a) Let $X_t = t\mathbf{x} \in \mathbb{R}^d$ for all $t \in [0, 1]$. Calculate $\mathbf{Sig}_{[0,1]}(X)$.

(b) Let $X \in \mathcal{C}_0^1([0, T], E)$ and $X_0 = 0$. Prove that

$$\mathbf{Sig}_{[0,1]}(X)_{1,2} + \mathbf{Sig}_{[0,1]}(X)_{2,1} = \mathbf{Sig}_{[0,1]}(X)_1 \cdot \mathbf{Sig}_{[0,1]}(X)_2.$$

(c) Let $X \in \mathcal{C}_0^1([0, 1], \mathbb{R})$ s.t. $X_t = \sin(t)$ for all $t \in [0, 1]$. Calculate $\mathbf{Sig}_{[0,1]}^{(2)}(X)$ i.e. the signatures of X up to order 2.

(d) Let $X \in \mathcal{C}_0^1([0, 1], \mathbb{R}^2)$ s.t. $X_t = (t, \sin(t))$ for all $t \in [0, 1]$. Calculate $\mathbf{Sig}_{[0,1]}^{(2)}(X)$ i.e. the signatures of X up to order 2.

(e) Let $X \in \mathcal{C}_0^1([0, 1], \mathbb{R})$ and $n \in \mathbb{N}$. Calculate $\int_0^1 t^n dX_t$ when

1. $X_t = t$
2. $X_t = \sin(t)$

Exercise 4.2 (Ito's formula) Let W be a Brownian motion on $[0, \infty)$ and define

$$Q^n(W) = \sum_{i=1}^n (W_{\frac{i}{n}} - W_{\frac{i-1}{n}})^2.$$

(a) Prove that $Q^n(W)$ converges to 1 in $\mathbb{L}^2$. Does the same property hold for a smooth ($\mathcal{C}^1$) function? What does this imply on the regularity of the paths of a Brownian motion W?

(b) Prove the following convergence in L^2 sense

$$\lim_{n\to\infty} \sum_{i=1}^n W_{\frac{i-1}{n}} (W_{\frac{i}{n}} - W_{\frac{i-1}{n}}) = \frac{W_1^2 - 1}{2}$$

(c) Prove that if f is smooth and bounded

$$f(W_t) = f(0) + \int_0^t f'(W_s) dW_s + \int_0^t \frac{f''(W_s)}{2} ds.$$

Exercise 4.3 (Black-Scholes model) Let $\sigma > 0$, $X_t = X_0 \exp\{\sigma W_t - \frac{\sigma^2 t}{2}\}$.

(a) Prove that X is a solution of

$$dX_t = \sigma X_t dW_t.$$

(b) Let $K > 0$, calculate

$$C_0 = \mathbb{E}[(X_T - K)_+].$$

(c) Let $K > 0$, calculate
$$\frac{\partial}{\partial X_0}\mathbb{E}[(X_T - K)_+].$$

Exercise 4.4 (Options' pricing in Black-Scholes and Heston models)

(a) Code option pricing and simulation for European call options and Digital call options in a Black-Scholes model. See exercise notebook 1.

(b) Compare simulations based on BS and Heston model. See exercise notebook 1.