Mathematics for New Technologies in Finance

Exercise sheet 5

Exercise 5.1 (Self financing portfolio) Recall the definition of self financing trading strategy ξ and its associated discounted value process $V = (V_t)_{t=0,\dots,T}$ is given by

$$V_0 := \xi_1 \cdot X_0 \quad \text{and} \quad V_t := \xi_t \cdot X_t \quad \text{for } t = 1, \dots, T.$$

The gains process associated with ξ is defined as

$$G_0 := 0 \quad \text{and} \quad G_t := \sum^{t} \xi_k \cdot (X_k - X_{k-1}) \quad \text{for } t = 1, \dots, T$$

(a) Prove $\xi_t \cdot X_t = \xi_{t+1} \cdot X_t$ for $t = 1, ..., T-1$.

(b) Prove $V_t = V_0 + G_t = \xi_1 \cdot X_0 + \sum_{k=1}^{t} \xi_k \cdot (X_k - X_{k-1})$ for all t.

Exercise 5.2 (Backpropogation of neural network) Let $\theta = (w, b, a) \in \mathbb{R}^3$ and let σ be the activation function. We consider the shallow neural network $f_\theta \colon \mathbb{R} \to \mathbb{R}$ s.t.

$$f_\theta(x) = a\sigma(wx + b).$$

Then we solve the regression problem with 3 data point $(x_i, y_i) \in \mathbb{R}^2$, $i = 1, 2, 3$ by minimizing the L^2 loss

$$\mathcal{L}_f = \sum_{i=1,2,3} (f_\theta(x_i) - y_i)^2.$$

(a) When solving the regression, do we compute $\nabla_{x_0} \mathcal{L}_f$ or $\nabla_\theta \mathcal{L}_f$?

(b) Compute $\partial_w f$ and $\partial_b f$ by chain rule. Do you find any intermediate value computed twice in both $\partial_w f$ and $\partial_b f$?

(c) Consider regression problem as a constrained optimization problem

$$\begin{aligned} &\min \sum_{i=1,2,3} l_i \\ &l_i = (\tilde{y}_i - y_i)^2 \\ &\tilde{y}_i = a\sigma(z_i), \qquad\qquad i = 1, 2, 3. \\ &z_i = wx_i + b \end{aligned}$$

Solve it by Lagrange multiplier and relate this with backpropagation.

(d) Generalize this idea to deep neural networks.

Exercise 5.3 (Backpropagation and cODEs) Translate a one layer neural network to a controlled ODE:

$$L^{(i)} : x \mapsto W^{(i)}x + a^{(0)} \mapsto \phi(W^{(1)}x + a^{(0)}),$$

with a cadlag control $u(t) = 1_{[1,2)}(t) + 2_{[2,\infty)}(t)$ and a time-dependent vector field

$$V(t, x) = 1_{[0,1)}(t) \left(L^{(0)}(x) - x \right) + 1_{[1,\infty)}(t) \left(L^{(1)}(x) - x \right).$$

The corresponding neural network at 'time' 3 is

$$x \mapsto L^{(0)}(x) \mapsto \phi(W^{(1)} L^{(0)}(x) + a^{(1)}).$$

(a) What is the evolution operator $J_{s,3}$?

(b) Calculate the derivative of the network with respect to parameters $W^{(1)}$ and $a^{(1)}$.

Exercise 5.4 (Hedging) See notebook 1.

Exercise 5.5 (Path-depedent derivatives in a BS market) See notebook 2.