Mathematics for New Technologies in Finance

Exercise sheet 7

Exercise 7.1 (Bayesian approach to implied volatility) The Black-Scholes formula provides a relationship between the price of a European Call option $C(K,T)$ and volatility $\sigma(K,T)$ for fixed price of underlying S_0, strike K, and maturity T. It is an important transformation in Finance to calculate from $C(K,T)$ the *implied volatility* $\sigma(K,T)$. Proceed in the following steps:

- Define a Gamma prior on implied volatility.
- Define a likelihood, which predicts the price given an implied volatility.
- Construct a posterior via Bayes formula and sample from it via Langevin dynamics. Interpret the resulting algorithm from the perspective of stochastic gradient descent.

Exercise 7.2 (NN for implied volatility) Recall the calculation of Implied volatility using Bayes formula from Exercise 1. Now we want to calculate the *implied volatility* $\sigma(K,T)$ from $C(K,T)$ using neural network. Proceed in the following steps:

- Define a neural network f^θ which takes as input the option price $C(K,T)$, the current price S_0, the strike price K, and the maturity T. The output will be the implied volatility $\sigma(K,T)$.
- Define a loss function L which calculates the difference between the actual price $C(K,T)$ and $f^\theta(C(K,T), S_0, K, T)$ inserted in the Black-Scholes formula.
- Run a gradient descent.

Exercise 7.3 (Breeden-Litzenberger formula)

(a) Is there always a positive implied volatility σ_{imp} related to the option price? If yes, prove it. Otherwise, on which price interval there is always a positive implied volatility σ_{imp} related to the option price?

(b) Prove the Breeden-Litzenberger formula:

$$\partial_K^2 C(T,K)dK = \mathrm{law}(S_T)(dK).$$

(c) Discretize the Breeden-Litzenberger formula and link it with Butterfly spreads.

Exercise 7.4 (Dupire formula) Assume the following local volatility model:

$$dS_t = \sigma(t,S_t)S_t dW_t.$$

(a) If $\sigma(t,S_t) = \sigma S_t^\beta$, for which value of β, the market has leverage effect (the volatility increases when the stock price goes down), which is empirically observed.

(b) Let V_t be the fair price of an European payoff $h(S_T)$. Prove the backward Kolmogorov equation:

$$\partial_t V_t + \frac{1}{2}\sigma(S,t)^2 S^2 \partial_{SS}^2 V_t = 0$$

(c) Let f_T^S be the probability density function of S_T, prove the forward Kolmogorov equation (Fokker-Planck equation):

$$\partial_T f(S,T) = \frac{1}{2}\partial_S^2\Big(\sigma(S,T)^2 S^2 f(S,T)\Big)$$

(d) Prove by Fokker-Planck equation the Dupire formula:

$$\sigma^2(K,T) = \frac{\partial_T C(T,K)}{\frac{1}{2}K^2\partial_K^2 C(T,K)}$$

where $C(T,K)$ is the European call option price of maturity T and strike K.