Mathematics for New Technologies in Finance

Solution sheet 10

Through this exercise sheet, we let $E = \mathbb{R}^d$, J an interval on $\mathbb{R}$, and denote $\mathbf{Sig}_J : \mathcal{C}_0^1(J, E) \to \mathbf{T}((E))$ the signature map such that for all $X \in \mathcal{C}_0^1(J, E)$ and we let $\mathbf{Sig}_J^{(M)}$ denote the truncated signature map up to order M: $\mathbf{Sig}_J^{(M)}(X) = (1, \mathbf{s}_1, \cdots, \mathbf{s}_M) \in \mathbf{T}^{(M)}((E))$. Let $X \in \mathcal{C}_0^1([0, s], E)$ and $Y \in \mathcal{C}_0^1([s, t], E)$. The concatenated path $X \star Y \in \mathcal{C}_0^1([0, t], E)$ is defined by

$$(X \star Y)_u = \begin{cases} X_u & u \in [0, s] \\ Y_u + (X_s - Y_s) & u \in [s, t] \end{cases}$$

Exercise 10.1 (Properties of signatures)

(a) (Invariance under reparametrization) Let $X \in \mathcal{C}_0^1([S_1, T_1], E)$ and $\tau : [S_2, T_2] \to [S_1, T_1]$ a $\mathcal{C}^1$, non-decreasing, surjective reparametrization. Then

$$\mathbf{Sig}_{[S_2,T_2]}(X_{\tau(\cdot)}) = \mathbf{Sig}_{[S_1,T_1]}(X).$$

(b) Prove that neither is $\mathbf{Sig}_{[0,1]}$ surjective nor the range of which a linear subspace of $\mathbf{T}((E))$.

(c) Prove that signature of the augmented paths i.e. $\bar{X}_t = (t, X_t)$ is unique.

(d) Prove that signatures satisfy the following equation

$$d\mathbf{Sig}_{[0,t]}(X) = \mathbf{Sig}_{[0,t]}(X) \otimes dX_t. \tag{1}$$

Solution 10.1

(a) We recall that, by definition,

$$\mathbf{Sig}_{[S,T]}(X) := 1 + \sum_{m=1}^{\infty} \sum_{i_1,\dots,i_m} (e_{i_1} \otimes \cdots \otimes e_{i_m}) \int_{S<u_1<\cdots<u_m<T} dX_{u_1}^{i_1}, \cdots, dX_{u_m}^{i_m},$$

where the integrals are understood as a Lebesgue–Stieltjes integral. Hence, one has only to prove that

$$\int_{S_2<u_1<\cdots<u_m<T_2} d(X^{i_1} \circ \tau)(u_1) \ \cdots \ d(X^{i_m} \circ \tau)(u_m) \ = \ \int_{S_1<v_1<\cdots<v_m<T_1} dX^{i_1}(v_1) \ \cdots \ dX^{i_m}(v_m).$$

To do so, one has simply to use the chain rule $d(X^i \circ \tau)(u) = (X^i)'(\tau(u))\tau'(u)du$, combined with the change of variables formula. Then, we write

$$\begin{aligned} &\int_{S_2<u_1<\cdots<u_m<T_2} d(X^{i_1} \circ \tau)(u_1) \cdots d(X^{i_m} \circ \tau)(u_m) \\ &\quad = \int_{\S_2<u_1<\cdots<u_m<T_2} \Big(\prod_{j=1}^{m} \tau'(u_j)\Big)\, dX^{i_1}(\tau(u_1)) \cdots dX^{i_m}(\tau(u_m))\, du_1 \cdots du_m \\ &\quad = \int_{S_1<v_1<\cdots<v_m<T_1} dX^{i_1}(v_1) \cdots dX^{i_m}(v_m), \end{aligned}$$

where we set $v_j = \tau(u_j)$ and notice that $\prod_{j=1}^{m} \tau'(u_j)$ corresponds to the Jacobian of the change of variables.

(b) We recall that, from Exercise 4.1 (b), it holds that

$$\mathbf{Sig}_{[0,1]}(X)_{1,2} + \mathbf{Sig}_{[0,1]}(X)_{2,1} = \mathbf{Sig}_{[0,1]}(X)_1 \cdot \mathbf{Sig}_{[0,1]}(X)_2. \tag{2}$$

This means that signatures must satisfy a shuffle identity (or group-like identity) on the tensor space $\mathbf{T}((E))$. Take now any element $w \in T((E))$, which we write

$$w \;=\; 1 + \sum_i a_i\, e_i + \sum_{i,j} b_{ij}\, (e_i \otimes e_j) + \cdots,$$

and notice that $w = \mathbf{Sig}_{[0,1]}(X)$ implies, due to (2), that

$$b_{ij} + b_{ji} \;=\; a_i\, a_j \quad \forall\, i,j.$$

Such a condition is obviously not satisfied for any w. Hence, the map is not surjective. For the same reason, the range cannot be a linear subspace of $\mathbf{T}((E))$. Given two paths X and Y, there is in general no reason for the relationship

$$\begin{aligned}
&[\mathbf{Sig}_{[0,1]}(X) + \mathbf{Sig}_{[0,1]}(Y)]_{1,2} + [\mathbf{Sig}_{[0,1]}(X) + \mathbf{Sig}_{[0,1]}(Y)]_{2,1} \\
&\mathbf{Sig}_{[0,1]}(X)_{1,2} + \mathbf{Sig}_{[0,1]}(Y)_{1,2} + \mathbf{Sig}_{[0,1]}(X)_{2,1} + \mathbf{Sig}_{[0,1]}(Y)_{2,1} \\
&\quad = [\mathbf{Sig}_{[0,1]}(X) + \mathbf{Sig}_{[0,1]}(Y)]_1 \cdot [\mathbf{Sig}_{[0,1]}(X) + \mathbf{Sig}_{[0,1]}(Y)]_2
\end{aligned}$$

to hold.

(c) It is a critical result in signature theory (also known as Hambly–Lyons theorem), the fact that the signature of a path is unique under tree like equivalent. Loosely speaking, tree-like equivalence means that two paths are regarded as “the same” if one can be obtained from the other by inserting or deleting pieces that “go out and back along the same track”so with one. The result can be intuitively proved noticing the full signature provides information on all the geometric moments of the path of X. Now, since $\bar{X}^0_t = t$ is strictly increasing, $\bar{X}$ cannot traverse any tree-like loop (which would require the time coordinate to backtrack) and therefore the only tree like equivalent path is the augmented path it self .

(d) Since X is $\mathcal{C}^1$, each iterated-integral depends smoothly on t. By the Leibniz rule, we get

$$\begin{aligned}
&\frac{d}{dt}\Big(\int_0^t \cdots \int_0^{t_m} dX^{i_m}_{t_m} \dots dX^{i_1}_{t_1}\Big)_{(i_1,\dots,i_m)\in\{1,\dots,d\}^m} \\
&= \Big(\int_0^{t_2} \cdots \int_0^{t_m} dX^{i_m}_{t_m} \dots dX^{i_2}_{t_2}\Big)_{(i_2,\dots,i_m)\in\{1,\dots,d\}^{m-1}} \Big(\frac{d}{dt} X^{i_1}_t\Big)_{i_1\in\{1,\dots,d\}}
\end{aligned}$$

Collecting now all the truncated signatures together, for the full signature it holds that

$$d\mathbf{Sig}_{[0,t]}(X) = \mathbf{Sig}_{[0,t]}(X) \otimes dX_t.$$

Hence, $\mathbf{Sig}_{[0,t]}(X)$ satisfies a linear ODE, with starting point $\mathbf{Sig}_{[0,0]}(X) = 1$.

Exercise 10.2 (Chen’s identity)

(a) Prove the Chen’s identity:

$$\mathbf{Sig}_{[r,t]}(X \star Y) = \mathbf{Sig}_{[r,s]}(X) \otimes \mathbf{Sig}_{[s,t]}(Y).$$

(b) Prove the Chen’s identity for truncated signature:

$$\mathbf{Sig}^{(M)}_{[r,t]}(X \star Y) = \mathbf{Sig}^{(M)}_{[r,s]}(X) \otimes \mathbf{Sig}^{(M)}_{[s,t]}(Y).$$

(c) Let X be linear on $[n, n+1]$ and let $X_{n+1} - X_n = \mathbf{x}_n$ for $n \in \mathbb{N}$, use Chen's identity to prove that

$$\mathbf{Sig}_{[0,N]}(X) = \bigotimes_{n \leq N} (1, \mathbf{x}_n, \frac{\mathbf{x}_n^{\otimes 2}}{2!}, \cdots).$$

Solution 10.2

(a) As seen in point (d) of the exercise above, $\mathbf{Sig}_{[0,t]}(X)$ satisfies a linear ODE. By the uniqueness of the solution of linear ode, we only need to check that $Z_u := \mathbf{Sig}_{[r,s]}(X) \otimes \mathbf{Sig}_{[s,u]}(Y)$ follows the dynamic of $\mathbf{Sig}_{[r,u]}(X \star Y)$, for any $u \in [r,t]$. At $u = r$, both sides equal the empty word:

$$\mathbf{Sig}_{[r,r]}(X \star Y) = 1 = \mathbf{Sig}_{[r,r]}(X).$$

Hence, we are only left to prove

$$\frac{d}{du} Z_u = Z_u \otimes (d(X \star Y)_u), \quad u \in [r,t].$$

For $u \in [r,s]$, this is obvious since $(X \star Y)_u = X_u$ and $\mathbf{Sig}_{[u,r]}(Y) = 1$. On the other interval $u \in [s,t]$, we notice first that $(X \star Y)_u = X_s + (Y_u - Y_s)$, so $d(X \star Y)_u = dY_u$. Since $\mathbf{Sig}_{[r,s]}(X)$is constant on $[s,t]$,

$$\frac{d}{du} Z_u = \mathbf{Sig}_{[r,s]}(X) \otimes \frac{d}{du} \mathbf{Sig}_{[s,u]}(Y) = \mathbf{Sig}_{[r,s]}(X) \otimes (\mathbf{Sig}_{[s,u]}(Y) \otimes dY_u) = Z_u \otimes d(X \star Y)_u,$$

where we used (1). At $u = t$, we get the required equality.

(b) We make again use of (1): since projection respects the linear ODE structure, we get that $\mathbf{Sig}^{(M)}_{[0,t]}(X)$ is a solution to the linear ODE

$$dH_t = H_t \otimes dX_t,$$

with $H_0 = 1$. This is nothing else that a ordinary (finite-dimensional) linear differential equation. By classical existence and uniqueness theory (Cauchy–Lipschitz theorem), such an equation admits a unique solution, which must therefore correspond to $\mathbf{Sig}^{(M)}_{[0,t]}(X)$. From this point on, one can follow the proof presented in point (a) to conclude.

(c) We already observed in Exercise 4.1 (a) that on each linear segment we have

$$\mathbf{Sig}_{[\frac{n}{N}, \frac{n+1}{N}]}(X) = (1, \mathbf{x}_n, \frac{\mathbf{x}_n^{\otimes 2}}{2!}, \cdots).$$

Then, it follows immediately from Chen's theorem that we have

$$\mathbf{Sig}_{[0,N]}(X) = \bigotimes_{n \leq N} (1, \mathbf{x}_n, \frac{\mathbf{x}_n^{\otimes 2}}{2!}, \cdots).$$